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▶ To cite this version:

Luigi Santocanale. Derived Semidistributive Lattices. Algebra Universalis, 2010, 63 (2), pp.101–130. 10.1007/s00012-010-0073-4 . hal-00165699v2

HAL Id: hal-00165699
https://hal.science/hal-00165699v2

Submitted on 21 Jul 2008

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Derived Semidistributive Lattices

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Abstract

For L a finite lattice, let $\mathbb{C}(L) \subseteq L^2$ denote the set of pairs $\gamma = (\gamma_0, \gamma_1)$ such that $\gamma_0 \prec \gamma_1$ and order it as follows: $\gamma \leq \delta$ iff $\gamma_0 \leq \delta_0$, $\gamma_1 \not\leq \delta_0$, and $\gamma_1 \leq \delta_1$. Let $\mathbb{C}(L, \gamma)$ denote the connected component of γ in this poset. Our main result states that, for any γ, $\mathbb{C}(L, \gamma)$ is a semidistributive lattice if L is semidistributive, and that $\mathbb{C}(L, \gamma)$ is a bounded lattice if L is bounded.

Let $\mathcal{S}_n$ be the Permutohedron on n letters and let $\mathcal{T}_n$ be the Associahedron on $n+1$ letters. Explicit computations show that $\mathbb{C}(\mathcal{S}_n, \alpha) = \mathcal{S}_{n-1}$ and $\mathbb{C}(\mathcal{T}_n, \alpha) = \mathcal{T}_{n-1}$, up to isomorphism, whenever α_1 is an atom of $\mathcal{S}_n$ or $\mathcal{T}_n$.

These results are consequences of new characterizations of finite join-semidistributive and of finite lower bounded lattices: (i) a finite lattice is join-semidistributive if and only if the projection sending $\gamma \in \mathbb{C}(L)$ to $\gamma_0 \in L$ creates pullbacks, (ii) a finite join-semidistributive lattice is lower bounded if and only if it has a strict facet labelling. Strict facet labellings, as defined here, are a generalization of the tools used by Barbut et al. [4] to prove that lattices of finite Coxeter groups are bounded.

1 Introduction

The set of covers of a finite lattice comes with a natural ordering induced by perspectivity. A cover of a lattice L is an ordered pair $\gamma = (\gamma_0, \gamma_1) \in L^2$ such that the interval $[\gamma_0, \gamma_1] = \{ x \in L \mid \gamma_0 \leq x \leq \gamma_1 \}$ is the two elements set $\{\gamma_0, \gamma_1\}$. As usual, we write $\gamma_0 \prec \gamma_1$ to mean that (γ_0, γ_1) is a cover. Two intervals $[x, y]$ and $[z, w]$ are perspective if either $x = y \wedge z$ and $w = y \vee z$, or, vice-versa, $z = x \wedge w$ and $y = x \vee w$. We order covers as follows: $\gamma \leq \delta$ if $\gamma_0 = \gamma_1 \wedge \delta_0$ and $\delta_1 = \gamma_1 \vee \delta_0$. Thus two covers are comparable if and only if they give rise to perspective intervals. The resulting poset, denoted here by $\mathbb{C}(L)$, is the object investigated in this paper.

The main result we shall present is that, whenever L is a finite semidistributive lattice, the poset of covers $\mathbb{C}(L)$ is the disjoint union of connected components each of which is again a semidistributive lattice; if moreover L is a bounded lattice in the sense of [14], then each such component is a bounded lattice as well. If γ is a cover of L, then we shall denote by $\mathbb{C}(L, \gamma)$ the connected component of γ in $\mathbb{C}(L)$ and call it the lattice derived from L by means of γ. Thus, if L is semidistributive, this process of constructing derivatives may be iterated.

These results are consequences of new characterizations of finite join-semidistributive lattices and of finite lower bounded lattices that strengthen well known facts. We remark here that, throughout this paper, we shall be interested in finite lattices only. For this reason and unless explicitly stated, the word lattice shall be a synonym of finite lattice.

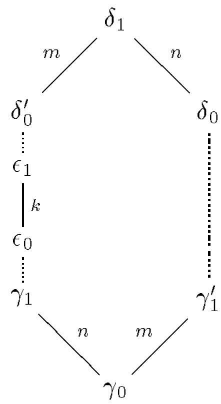
On one side, it is well known that a lattice is join-semidistributive if and only if, given a cover $\gamma_0 \prec \gamma_1$, there exists a unique cover $j_* \prec j$, perspective to $\gamma_0 \prec \gamma_1$, such that j is join-irreducible [8, §2.56]. The latter property may be rephrased by saying that for each $\gamma \in \mathbb{C}(L)$ there exists a unique $\iota \in \mathbb{C}(L)$, minimal within $\mathbb{C}(L)$, such that ι and γ are comparable. Following a suggestion of [4, Theorem 1], we observe that such uniqueness property is consequence of the pushdown relation $\rightharpoondown$ between covers being confluent.[1] This relation shall be introduced in Section 3; by studying further it, we refine the existing characterization of join-semidistributivity to the following statement: *a lattice is join-semidistributive if and only if the poset of covers has pullbacks.*

On the side of lower bounded lattices, we build on the ideas used in [4] to prove that lattices arising from Cayley graphs of finite Coxeter groups are bounded. With respect to that work, we move from a sufficient condition to a complete characterization, and from boundedness to the weaker notion of lower boundedness. The statement reads as follows: *a lattice is lower bounded if and only if it is join-semidistributive and has a strict facet labelling.* A strict facet labelling is a labelling of covers by natural numbers subject to some constraints. We illustrate next these constraints under the simplifying assumption that L is a semidistributive lattice. As illustrated in Figure 1, a facet[2] is a quadruple of covers $\delta, \delta', \gamma, \gamma' \in \mathbb{C}(L)$ such that $\gamma_0 = \gamma'_0 = \delta_0 \wedge \delta'_0 < \gamma_1 \vee \gamma'_1 = \delta_1 = \delta'_1$, with $\gamma_1 \leq \delta'_0$ and $\gamma'_1 \leq \delta_0$. For such a facet, we shall prove that $\gamma \prec \delta$ (as well as $\gamma' \prec \delta'$) in the poset $\mathbb{C}(L)$ and, moreover, that every cover of covers arises from a facet. A strict facet labelling assigns the same number to γ and δ and, consequently, it is constant on connected components of $\mathbb{C}(L)$. Moreover, such a labelling is required to be strictly increasing at the interior of a facet which is a pentagon: if $\epsilon \in \mathbb{C}(L)$ is such that $\gamma_1 \leq \epsilon_0 \prec \epsilon_1 \leq \delta'_0$, then δ_0, δ'_0, and γ_1 generate a sublattice which is a pentagon; then the label of ϵ should be striclty greater than the labels of γ and of δ'.

We shall carry out some explicit computations: we prove that for the Permutohedron $\mathcal{S}_n$ – i.e. the lattice of permutations on n letters with the weak

[1] Let us recall that a relation $\rightarrow \subseteq V \times V$ is confluent if $v_0 \rightarrow^* v_i$, $i = 1, 2$, implies that $v_i \rightarrow^* v_3$, $i = 1, 2$ for some $v_3 \in V$; here $\rightarrow^*$ denotes the reflexive transitive closure of $\rightarrow$. If the relation $\rightarrow$ has no infinite path, then it is a standard result that for each v there exists a unique v' such that $v \rightarrow^* v'$ and v' has no successor.

[2] In [4] a facet is called a 2-facet, using a more precise wording from combinatorial geometry.



$m, n < k$.

Figure 1: A strict labelling of a facet

Bruhat order – and the Associahedron $\mathcal{T}_n$ – the Tamari lattice on $n+1$ letters – the relations

$$\mathbb{C}(\mathcal{S}_n\,, \alpha) = \mathcal{S}_{n-1}\,, \qquad\qquad \mathbb{C}(\mathcal{T}_n\,, \alpha) = \mathcal{T}_{n-1}\,,$$

hold up to isomorphism, whenever α is of the form $(\perp, \alpha_1)$, that is, α_1 is an atom. A similar relation holds for $\mathcal{B}^n$, the Boolean algebra on n atoms. Since in $\mathcal{B}^n$ the join irreducible elements are exactly the atoms, the property is there stronger in that $\mathbb{C}(\mathcal{B}^n\,, \gamma) = \mathcal{B}^{n-1}$ *for every* $\gamma \in \mathbb{C}(\mathcal{B}^n\,)$. As a consequence

$$\mathbb{C}(\mathcal{B}^n\,) = \coprod_{i=1,\ldots,n} \mathcal{B}^{n-1} = n \cdot \mathcal{B}^{n-1}\,, \tag{1}$$

where the coproduct and the products – implicit in the exponents i of $\mathcal{B}^i$ – are taken within the category of posets and order preserving functions. A reason for the name of derived lattices becomes transparent if, in equation (1), we replace the symbol $\mathbb{C}$ with the symbol $\frac{\partial}{\partial \mathcal{B}}$.

It is suggestive to call the lattices $\mathcal{S}_n, \mathcal{T}_n$, and $\mathcal{B}^n$ *regular*, meaning that the shape of $\mathbb{C}(L, \alpha)$ does not depend on the choice of the atom α_1. This use of terminology from combinatorial geometry is on purpose since, with this work, we aim at giving a ground to some intuitions relating algebra, order, and geometry. In combinatorial geometry a Permutohedron is a particular convex polytope [21, 0.10]. By orienting the graph reduct of this polytope we obtain the Hasse diagram of the lattice of permutations. We shall show that the 2-facets of this polytope can also be oriented so that they give rise to the Hasse diagram of $\mathbb{C}(\mathcal{S}_n\,)$, i.e. to covers of covers, or say 2-covers. By semidistributivity, we can define n-covers as elements of $\mathbb{C}^n\,(\mathcal{S}_n)$. It is not difficult to verify that these correspond to oriented n-facets of the polytope $\mathcal{S}_n$.

Since we can define n-facets for an arbitrary semidistributive lattice, it becomes natural to ask whether the lattice theoretic algebra plays any role in geometry and, for example, whether we can characterize lattices that arise from convex polytopes such as $\mathcal{S}_n, \mathcal{T}_n$, and $\mathcal{B}^n$ by lattice theoretic means. This paper will not answer these questions, its purpose being limited to settling a ground for future researches.

Let us also stress on the fact that ideas and results presented here have their origin at the intersection between order theory and the theory of rewriting systems, in the spirit of [16] and [2]. Even if we are not going to emphasize this aspect, it is worth recalling it. It was suggested in [4] that the perspectivity relation between covers is generated by a sort of rewriting system. In section 3 we make explicit such a rewriting system, namely, we define the pushdown relation $\rightharpoondown$ between covers. We also explicitly introduce the class of pushdown lattices, as our proofs make heavy use its properties. Briefly, a lattice is pushdown if the perspectivity order on covers is generated by the pushdown relation, thus allowing a sort of local reasoning on the global structure of the covers.

The paper is structured as follows. We first recall the basic concepts concerning lattices in Section 2 and we introduce pushdown lattices in Section 3. We give in Section 4 our characterization of join-semidistributive lattices, and in Section 5 our characterization of lower bounded lattices. We define in Section 6 lattices of the form $\mathbb{C}(L, \gamma)$, that is lattices derived from semidistributive lattices, and prove then that properties such as semidistributivity and being lower bounded lift from L to $\mathbb{C}(L, \gamma)$. Finally, in Section 7, we exemplify the construction of derived lattices on Newman lattices.

2 Preliminaries

We begin by introducing standard definitions and notations on finite lattices.

Let P be a poset. A *cover* in P is an ordered pair $(\gamma_0, \gamma_1) \in P^2$ such that $\gamma_0 \leq \gamma_1$ and the closed interval $\{ x \in L \mid \gamma_0 \leq x \leq \gamma_1 \}$ is the two element set $\{ \gamma_0, \gamma_1 \}$ – in particular $\gamma_0 \neq \gamma_1$. As usual, we shall write $\gamma_0 \prec \gamma_1$ if (γ_0, γ_1) is a cover and say that γ_0 is a lower cover of γ_1 and that γ_1 is an upper cover of γ_0. We shall denote by $\mathbb{C}(P)$ the set of covers of P and use Greek letters $\gamma, \delta \ldots$ to denote these covers.

An element of a lattice L is *join-*(resp. *meet-*)*irreducible* if it has a unique lower (resp. upper) cover. We denote by $J(L)$ (resp. $M(L)$) the set of join-(resp. meet-)irreducible elements of L. If $j \in J(L)$ then j_* denotes the unique element of L such that $j_* \prec j$. If $m \in M(L)$, then m^* denotes the unique element of L such that $m \prec m^*$. Let us introduce the standard arrow relations between join-irreducible and meet-irreducible elements. For $j \in J(L)$ and $m \in M(L)$, we write $j \uparrow m$ if $j \leq m^*$ and $j \not\leq m$, and $j \downarrow m$ if $j_* \leq m$ and $j \not\leq m$; we write $j \updownarrow m$ if $j \uparrow m$ and $j \downarrow m$.

We finally recall that a lattice is *join-semidistributive* if it satisfies the Horn sentence

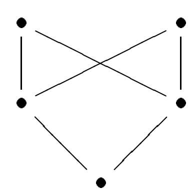
$$x \vee y = x \vee z \Rightarrow x \vee (y \wedge z) = x \vee y\,. \qquad (SD_\vee)$$

A lattice is *meet-semidistributive* if it satisfies the Horn sentence dual of $(SD_\vee)$. It is *semidistributive* if it is both meet-semidistributive and join-semidistributive.

Posets with pullbacks. These posets will play a central role in our development. We introduce them now together with their elementary properties. Recall from [4] that a *hat* in a poset P is a triple (u, v, w) such that $u \prec v$, $w \prec v$, and $u \neq w$. An *antihat* in P is defined dually. A *cospan* in P is a triple of elements (u, v, w) such that $u \leq v$ and $w \leq v$; in particular a hat is a particular kind of a cospan. We say that a cospan (u, v, w) *has a pullback* if the meet $u \wedge w$ exists.

Definition 2.1. We say that a poset P *has pullbacks* if every cospan in P has a pullback. We say that $f : P \longrightarrow Q$ *preserves pullbacks* if $f(u \wedge w)$ is the pullback of the cospan $(f(u), f(v), f(w))$, whenever $u \wedge w$ is the pullback of the cospan (u, v, w).

Clearly, a poset has pullbacks iff every finite non empty set admitting an upper bound has a meet. Notice that every meet-semilattice is a poset with pullbacks. The following diagram exhibits a poset with pullbacks which is not a meet-semilattice.



The following Proposition is almost a reformulation of Definition 2.1.

Proposition 2.2. *A poset P has pullbacks iff every principal ideal of P is a lattice.*

We state next, without proofs, some facts illustrating the specific role of pullbacks of hats among all the pullbacks.

Lemma 2.3. *A finite poset P has pullbacks iff every hat has a pullback.*

Lemma 2.4. *Let P, Q be finite posets with pullbacks. If $f : P \longrightarrow Q$ preserves pullbacks of hats, then it preserves pullbacks.*

The following observations will turn to be more interesting for our goals.

Proposition 2.5. *If a finite poset P has pullbacks, then each connected component of P has a least element.*

Proof. Since a principal ideal is a finite lattice, then it has a least element. Let therefore $\perp_x$ denote the least element of the principal ideal of x. We argue that if x, y belong to the same connected component, then $\perp_x = \perp_y$. To this goal it suffices to establish that $\perp_x = \perp_y$ whenever $z \leq x, y$ for some z. We have $\perp_x \leq z \leq y$, and hence $\perp_y \leq \perp_x$. By symmetry, $\perp_x \leq \perp_y$. $\square$

Let us say that P *has pushouts* if its dual poset P^{op} has pullbacks.

Corollary 2.6. *If a finite poset P has pullbacks and pushouts, then each connected component of P is a lattice.*

Proof. Since P has pushouts each connected component has a maximum element, that is, it is a principal ideal. Since P has pullbacks, such an ideal is a lattice. $\square$

3 Pushdown Lattices

In the following L will denote a fixed lattice. Let us recall that an interval $[x, y]$ is said to *transpose down* to an interval $[z, w]$ if $z = x \wedge w$ and $y = x \vee w$. Two intervals $[x, y]$ and $[z, w]$ are said to be *perspective* if either $[x, y]$ transposes down to $[z, w]$, or $[z, w]$ transposes down to $[x, y]$. Perspectivity suggests how to define an ordering $\leq$ on the set $\mathbb{C}(L)$ of covers of L: for $\gamma, \delta \in \mathbb{C}(L)$, we let

$$\gamma \leq \delta \text{ if } \gamma_0 \leq \delta_0,\ \gamma_1 \not\leq \delta_0, \text{ and } \gamma_1 \leq \delta_1\,.$$

It is easy to verify that, for $\gamma, \delta \in \mathbb{C}(L)$, $\gamma \leq \delta$ iff $[\gamma_0, \gamma_1]$ and $[\delta_0, \delta_1]$ are perspective with $[\delta_0, \delta_1]$ transposing down to $[\gamma_0, \gamma_1]$. This is an order relation on $\mathbb{C}(L)$, since L is a lattice: if $\gamma \leq \delta \leq \epsilon$ then clearly $\gamma_i \leq \epsilon_i$ for $i = 0, 1$, and if $\gamma_1 \leq \epsilon_0$, then also $\gamma_1 \leq \delta_1 \wedge \epsilon_0 = \delta_0$, a contradiction. Observe also that a sufficient condition for such a relation to be an ordering is that L has pullbacks. When referring to the poset $\mathbb{C}(L)$ we shall mean the ordered pair $\langle \mathbb{C}(L), \leq \rangle$. By $\mathbb{C}(L, \gamma)$ we shall denote the connected component of γ in $\mathbb{C}(L)$.

The two projections

$$(\,\cdot\,)_i : \mathbb{C}(L) \longrightarrow L\,, \qquad\qquad i = 0, 1\,,$$

sending γ to γ_i, are order preserving. They will play a key role in the rest of the paper.

Let P, Q be posets, an order preserving map $\pi : P \longrightarrow Q$ is *conservative* if it strictly preserves the order, i.e. $x \leq y$ and $\pi(x) = \pi(y)$ imply $x = y$, or, equivalently $x < y$ implies $\pi(x) < \pi(y)$. Our first remark is the following:

Lemma 3.1. *In any lattice L the projections $(\,\cdot\,)_i$, $i = 0, 1$, are conservative.*

Proof. If $\gamma \leq \delta$ and $\gamma_0 = \delta_0$, then the relations $\delta_0 = \gamma_0 \prec \gamma_1 \leq \delta_1$ and $\delta_0 \prec \delta_1$ imply $\delta_1 = \gamma_1$. $\square$

If P, Q are two posets, then an order preserving function $\pi : Q \longrightarrow P$ is a *Grothendieck fibration* if for each $\delta \in Q$ the restriction of π to the principal ideal generated by δ is an embedding. Spelled out, this means that $\gamma, \epsilon \leq \delta$ and $\pi(\gamma) \leq \pi(\epsilon)$ implies $\gamma \leq \epsilon$.

Definition 3.2. We say that a lattice L is a *pushdown* lattice if the projection $(\,\cdot\,)_0 : \mathbb{C}(L) \longrightarrow L$ is a Grothendieck fibration.

We shall say that L is a *pushup* lattice if the dual L^{op} is pushdown. Spelled out, L is pushdown if the following property holds:

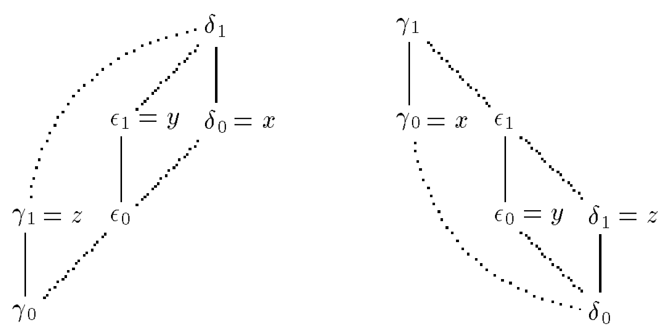
$$\gamma, \epsilon \leq \delta \text{ and } \gamma_0 \leq \epsilon_0 \text{ imply } \gamma \leq \epsilon\,.$$

Later, we shall refer to this property as to the *pushdown property.* Dually, L is pushup if $\delta \leq \gamma, \epsilon$ and $\epsilon_1 \leq \gamma_1$ imply $\epsilon \leq \gamma$. These properties may be simplified even more, for example L is pushdown iff $\gamma, \epsilon \leq \delta$ and $\gamma_0 \leq \epsilon_0$ imply $\gamma_1 \leq \epsilon_1$.

Examples of lattices enjoying these properties arise from some form of semidistributivity.

Lemma 3.3. *If L is a join-semidistributive lattice, then it is both a pushdown and a pushup lattice.*

Proof. The proof is sketched in the two diagrams below:



Let x, y, z be as above on the left: if $\gamma_1 \not\leq \epsilon_1$, then $y \wedge z = \epsilon_1 \wedge \gamma_1 = \gamma_0$ and, consequently, $x \vee y = x \vee z$ but $x \vee (y \wedge z) = x \vee \gamma_0 = x < \delta_1 = x \vee y$.

Let x, y, z be as above on the right: if $\gamma_0 \not\geq \epsilon_0$, then $x \vee y = \gamma_0 \vee \epsilon_0 = \gamma_1$ and, consequently, $x \vee y = x \vee z$ but $x \vee (y \wedge z) = x \vee \delta_0 = x < \gamma_1 = x \vee y$. $\square$

By duality, it follows that meet-semidistributive lattices are both pushdown and pushup lattices.

To understand the relationship between pushdown lattices and hats and antihats as defined in Section 2, let us introduce the *pushdown relation* $\rightharpoondown$ on $\mathbb{C}(L)$ as follows. For $u \in L$ and $\gamma, \delta \in \mathbb{C}(L)$, let us write $\delta \stackrel{u}{\rightharpoondown} \gamma$ if $u \neq \delta_0$, $\gamma_0 = u \wedge \delta_0$, and $\gamma_1 \leq u \prec \delta_1$. For $\gamma, \delta \in \mathbb{C}(L)$, let us write $\delta \rightharpoondown \gamma$ if $\delta \stackrel{u}{\rightharpoondown} \gamma$ for some $u \in L$. A first remark is that $\delta \rightharpoondown \gamma$ implies $\gamma < \delta$, in any lattice: if $\delta \stackrel{u}{\rightharpoondown} \gamma$, then $\gamma_0 = u \wedge \delta_0 \leq \delta_0$, $\gamma_1 \leq u \leq \delta_1$; if moreover $\gamma_1 \leq \delta_0$, then $\gamma_0 < \gamma_1 \leq u \wedge \delta_0 = \gamma_0$, a contradiction. Next, we observe that if $\delta \stackrel{u}{\rightharpoondown} \gamma$, then (u, δ_1, δ_0) is a hat and γ_0 is its pullback. The next Lemma implies that, in a pushdown lattice, to a given hat (u, δ_1, δ_0) there corresponds a unique antihat (x, γ_0, y) such that $\gamma_0 = u \wedge \delta_0$, $x \leq u$, and $y \leq \delta_0$.

Lemma 3.4. *In a pushdown lattice, for each $\delta \in \mathbb{C}(L)$ and $u \in L$ such that $u \prec \delta_1$ and $u \neq \delta_0$, there exists a unique γ such that $\delta \stackrel{u}{\rightharpoondown} \gamma$.*

Proof. Let $u \in L$ and $\delta \in \mathbb{C}(L)$ be as in the statement of the Lemma.

We first construct $\gamma \in \mathbb{C}(L)$ such that $\delta \xrightarrow{u} \gamma$. Let $\gamma_0 = u \wedge \delta_0$ and observe that $\gamma_0 < u$, since u, δ_0 are not comparable. Hence, we can choose γ_1 such that $\gamma_0 \prec \gamma_1 \leq u$ and define $\gamma = (\gamma_0, \gamma_1)$. It is easily verified that $\delta \xrightarrow{u} \gamma$.

Let us suppose next that $\delta \xrightarrow{u} \gamma$ and $\delta \xrightarrow{u} \epsilon$, we shall show that $\gamma = \epsilon$. By the definition of $\rightarrow$, $\gamma_0 = u \wedge \delta_0 = \epsilon_0$, hence $\gamma_0 \leq \epsilon_0$. Also $\delta \xrightarrow{u} \gamma$ implies $\gamma \leq \delta$ and, similarly, $\delta \xrightarrow{u} \epsilon$ implies $\epsilon \leq \delta$. Hence $\gamma, \epsilon \leq \delta$, $\gamma_0 \leq \epsilon_0$, and the pushdown property imply $\gamma \leq \epsilon$. By symmetry, we obtain $\epsilon \leq \gamma$ as well. $\square$

Proposition 3.5. *In a pushdown lattice L the order of $\mathbb{C}(L)$ is the converse relation of the reflexive transitive closure of $\rightarrow$.*

Proof. Let $\xrightarrow{*}$ denote the reflexive transitive closure of $\rightarrow$. We have already seen that $\delta \rightarrow \gamma$ implies $\gamma < \delta$ in any lattice, hence, if $\delta \xrightarrow{*} \gamma$, then $\gamma \leq \delta$ as well.

Therefore we shall focus on proving the converse implication, that is, if L is pushdown and $\gamma \leq \delta$, then we can find a path of the relation $\rightarrow$ from δ to γ. The proof is by induction on the height of the interval $[\gamma_0, \delta_0]$.

If $\gamma_0 = \delta_0$, then $\gamma \leq \delta$ and Lemma 3.1 imply $\gamma = \delta$. Thus, the empty path witnesses the relation $\delta \xrightarrow{*} \gamma$.

Let us suppose that $\gamma_0 < \delta_0$ so that, by Lemma 3.1, $\gamma_1 < \delta_1$ as well. Pick $u \in L$ such that $\gamma_1 \leq u \prec \delta_1$ and observe that $u \neq \delta_0$, since $\gamma_1 \not\leq \delta_0$. Let ϵ be determined by the property that $\delta \xrightarrow{u} \epsilon$. Then $\gamma_0 \leq u \wedge \delta_0 = \epsilon_0$, $\gamma, \epsilon \leq \delta$, and therefore, by the pushdown property, $\gamma \leq \epsilon$. Moreover, $[\gamma_0, \epsilon_0] \subset [\gamma_0, \delta_0]$ and, by the induction hypothesis, $\epsilon \xrightarrow{*} \gamma$. Considering that $\delta \rightarrow \epsilon \xrightarrow{*} \gamma$, it follows that $\delta \xrightarrow{*} \gamma$. $\square$

We end this section with a characterization of pushdown lattices in terms of the relation $\rightarrow$. This was actually our original definition of pushdown lattices: if $\delta \xrightarrow{u} \gamma$, then the cover δ has been lowered to the cover γ by means of the push of u. This intuition is already present in [4].

Proposition 3.6. *A lattice L is a pushdown lattice if and only if $\gamma \leq \delta$ and $\delta \xrightarrow{u} \epsilon$ with $\gamma_0 \leq u$ imply $\gamma \leq \epsilon$.*

Proof. Clearly, if a lattice is pushdown, $\gamma \leq \delta$, $\delta \xrightarrow{u} \epsilon$, and $\gamma_0 \leq u$, then $\gamma_0 \leq u \wedge \delta_0 = \epsilon_0$, $\gamma, \epsilon \leq \delta$, and hence $\gamma \leq \epsilon$.

Conversely, let us suppose that $\gamma, \epsilon \leq \delta$ with $\gamma_0 \leq \epsilon_0$ and that L has the property stated in the Proposition. The reader will have no difficulties to adapt the proof of Proposition 3.5 in order to construct a path from δ to ϵ of the form $\delta = \theta^0 \xrightarrow{u_1} \theta^1 \ldots \theta^{n-1} \xrightarrow{u_n} \theta^n = \epsilon$. By assumption we have $\gamma \leq \delta = \theta^0$. Let us suppose next that $\gamma \leq \theta^i$, for some $i \in \{0, \ldots, n-1\}$. Since $\theta^i \xrightarrow{u_{i+1}} \theta^{i+1}$ and $\gamma_0 \leq \epsilon_0 \leq u_{i+1}$, the property of the Proposition ensures that $\gamma \leq \theta^{i+1}$. We have therefore $\gamma \leq \theta^n = \epsilon$, showing that L is pushdown. $\square$

4 Join-Semidistributive Lattices

Many characterizations of (finite) join-semidistributive lattices are already available, see [1, Theorem 1.11] for an example. In this section we introduce one more characterization, Theorem 4.3. A closer glance to this Theorem shows that it is a refinement of a well known characterization stating that a lattice is join-semidistributive if and only if for each meet-irreducible element m there exists a unique join-irreducible element j such that $j \updownarrow m$. This characterization may be rephrased in terms of the poset $\mathbb{C}(L)$. The correspondence sending a meet-irreducible element m to the cover (m, m^*) establishes a bijection between $M(L)$ and the set of maximal elements of $\mathbb{C}(L)$. A similar bijection may be defined between $J(L)$ and the set of minimal elements of $\mathbb{C}(L)$. With these bijections at hand, we can state the previous characterization as follows:

Lemma 4.1 (See Theorem 2.56 in [8]). *A lattice L is join-semidistributive if and only if each maximal element of $\mathbb{C}(L)$ has a least element below it.*

Roughly speaking, the new characterizations is obtained from the previous one by replacing the statement *each maximal element of $\mathbb{C}(L)$ has a least element below it* with the more informative statement *$\mathbb{C}(L)$ has pullbacks.*

An order preserving function $\pi : P \longrightarrow Q$ *creates pullbacks* if the following condition holds: whenever $x, y, z \in P$ are such that $x, y \leq z$ and $\pi(x) \wedge \pi(y)$ exists in Q, then there exists a unique $u \leq x, y$ such that $\pi(u) = \pi(x) \wedge \pi(y)$; moreover $u = x \wedge y$. For conservative order preserving maps this condition splits as the conjunction of two conditions, as stated in the following Lemma.

Lemma 4.2. *A conservative order preserving function $\pi : P \longrightarrow Q$* creates pullbacks *if and only if (i) it is a Grothendieck fibration and (ii) if $x, y \leq z \in P$ and $\pi(x) \wedge \pi(y)$ exists in Q, then there exists $u \leq x, y$ such that $\pi(u) = \pi(x) \wedge \pi(y)$.*

Proof. Let us suppose that π creates pullbacks, so that (ii) certainly holds. If $x, y \leq z$ and $\pi(x) \leq \pi(y)$, then $\pi(x) = \pi(x) \wedge \pi(y)$, so that the meet of $\pi(x), \pi(y)$ exists in Q. Let $u \leq x, y$ be such that $\pi(u) = \pi(x) \wedge \pi(y) = \pi(x)$. Since π is conservative, then $u = x$, so that $x \leq y$, exhibiting π as a Grothendieck fibration.

Conversely, let us suppose that (i) and (ii) hold. Let $x, y \leq z$ be such that $\pi(x) \wedge \pi(y)$ exists, and let $u, u' \leq x, y$ be two preimages of this meet. Since π is an embedding when restricted to the principal ideal of z and $\pi(u) = \pi(u')$, then $u = u'$ as well. Let us show that $u = x \wedge y$: if $w \leq x, y$, then $\pi(w) \leq \pi(x), \pi(y)$ hence $\pi(w) \leq \pi(x) \wedge \pi(y) = \pi(u)$. Since $w, u \leq z$, we deduce $w \leq u$, since π is a Grothendieck fibration. $\square$

An obvious remark, worth recalling at this point, is that if an order preserving map $\pi : P \longrightarrow Q$ creates pullbacks and Q has pullbacks, then P has pullbacks as well which are preserved by π. We state next the main result of this section.

Theorem 4.3. *A lattice L is join-semidistributive if and only if the projection $(\,\cdot\,)_0 : \mathbb{C}(L) \longrightarrow L$ creates pullbacks.*

Proof. If $(\,\cdot\,)_0$ creates pullbacks then, by the previous remark, $\mathbb{C}(L)$ has pullbacks. If $\mu \in \mathbb{C}(L)$ is maximal, then the ideal it generates is a finite meet-semilattice and hence it has a least element. Therefore L is join-semidistributive.

Conversely, let us suppose that L is join-semidistributive. Since $(\,\cdot\,)_0$ is conservative and, by Lemma 3.3, a Grothendieck fibration, it is enough by Lemma 4.2 to show that if $\gamma, \delta \leq \epsilon$, then we can find $\beta \leq \gamma, \delta$ such that $\beta_0 = \gamma_0 \wedge \delta_0$.

Observe first that $\gamma_0 \wedge \delta_0 < \gamma_1 \wedge \delta_1$: otherwise, if $\gamma_0 \wedge \delta_0 = \gamma_1 \wedge \delta_1$, then $\epsilon_1 = \gamma_1 \vee \epsilon_0 = \delta_1 \vee \epsilon_0 = (\gamma_1 \wedge \delta_1) \vee \epsilon_0 = (\gamma_0 \wedge \delta_0) \vee \epsilon_0 = \epsilon_0$, a contradiction. Let $\beta_0 = \gamma_0 \wedge \delta_0$ and choose β_1 such that $\beta_0 \prec \beta_1 \leq \gamma_1 \wedge \delta_1$. We claim that $\beta_1 \not\leq \gamma_0$: otherwise, if $\beta_1 \leq \gamma_0$, then $\beta_1 \leq \epsilon_0 \wedge \delta_1 = \delta_0$ and $\beta_1 \leq \gamma_0 \wedge \delta_0 = \beta_0$ contradicting $\beta_0 \prec \beta_1$. Therefore $\beta_0 \leq \gamma_0$, $\beta_1 \not\leq \gamma_0$, and $\beta_1 \leq \gamma_1$, that is $\gamma \leq \beta$. We derive $\beta \leq \delta$ similarly. □

The rest of this section is devoted to characterizing join-semidistributive lattices among pushdown lattices.

Lemma 4.4. *If L is a pushdown lattice and $\gamma \prec \delta$ in the poset $\mathbb{C}(L)$, then $\delta \rightharpoondown \gamma$. If L is a join-semidistributive lattice and $\delta \rightharpoondown \gamma$, then $\gamma \prec \delta$ in $\mathbb{C}(L)$.*

Proof. Let us first assume that L is pushdown and that $\gamma \prec \delta$ in $\mathbb{C}(L)$. Obviously, $\gamma < \delta$ and therefore $\gamma_1 < \delta_1$, by Lemma 3.1. Let $u \in L$ be such that $\gamma_1 \leq u \prec \delta_1$ and observe that $u \neq \delta_0$, otherwise $\gamma_1 \leq u = \delta_0$ contradicting $\gamma \leq \delta$. Let $\epsilon \in \mathbb{C}(L)$ be the unique cover such that $\delta \overset{u}{\rightharpoondown} \epsilon$. We have $\gamma \leq \delta$ and $\gamma_0 \leq u$, hence $\gamma \leq \epsilon$ by Proposition 3.6. Therefore $\gamma \leq \epsilon < \delta$, so that $\gamma \prec \delta$ implies $\gamma = \epsilon$. As $\gamma = \epsilon$, then we deduce that $\delta \overset{u}{\rightharpoondown} \gamma$.

Next, we suppose that L is a join-semidistributive lattice and that $\delta \overset{u}{\rightharpoondown} \gamma$. To prove that $\gamma \prec \delta$, we consider $\epsilon \in \mathbb{C}(L)$ such that $\gamma \leq \epsilon \leq \delta$ and argue that either $\epsilon = \delta$, or $\epsilon \leq \gamma$. If $\epsilon_0 \not\leq u$, then $\epsilon_1 \not\leq u$ and $\epsilon_1 \vee u = \delta_1$. By join-semidistributivity, the relations $\epsilon_1 \vee u = \delta_1 = \epsilon_1 \vee \delta_0$ imply $\delta_1 = \epsilon_1 \vee (u \wedge \delta_0) = \epsilon_1 \vee \gamma_0 = \epsilon_1$. By Lemma 3.1, $\epsilon \leq \delta$ and $\epsilon_1 = \delta_1$ imply $\epsilon = \delta$. Otherwise, if $\epsilon_0 \leq u$, then Lemma 3.3 and Proposition 3.6 ensure that $\epsilon \leq \gamma$. □

Proposition 4.5. *For a pushdown lattice L the following are equivalent:*

1. *If $\delta \rightharpoondown \gamma$ then $\gamma \prec \delta$,*
2. *If $u \neq v$, $\delta \overset{u}{\rightharpoondown} \gamma$, and $\delta \overset{v}{\rightharpoondown} \epsilon$, then γ_0, ϵ_0 are not comparable.*
3. *Every hat in $\mathbb{C}(L)$ has a pullback.*
4. *$\mathbb{C}(L)$ has pullbacks.*
5. *L is join-semidistributive.*

Proof. (1) implies (2). Let us suppose that $u \neq v$, $\delta \overset{u}{\rightharpoondown} \gamma$, and $\delta \overset{v}{\rightharpoondown} \epsilon$.

By the way of contradiction, let us suppose that $\gamma_0 \leq \epsilon_0$. Then $\gamma, \epsilon < \delta$ and $\gamma_0 \leq \epsilon_0$ imply $\gamma \leq \epsilon$. Taking into account that, by (1), $\gamma \prec \delta$, then $\gamma \leq \epsilon < \delta$ implies $\gamma = \epsilon$. Consequently, $\delta \overset{u}{\rightharpoondown} \gamma$ and $\delta \overset{v}{\rightharpoondown} \gamma$ imply $\gamma_1 \leq u \wedge v$.

Next, let $\delta' = (u, \delta_1)$ and let $\psi, \xi \in \mathbb{C}(L)$ be the covers determined by the relations $\delta' \overset{\delta_0}{\rightharpoondown} \psi$ and $\delta' \overset{v}{\rightharpoondown} \xi$. Observe that $\psi_0 < \xi_0$, since $\psi_0 = u \wedge \delta_0 = \gamma_0 < \gamma_1 \leq u \wedge v = \xi_0$. Thus, considering that $\psi_0 < \xi_0$ and $\psi, \xi \leq \delta'$, we can use the pushdown property to deduce that $\psi < \xi$. However, this is a contradiction: we have $\psi < \xi < \delta'$ and, by (1), $\psi \prec \delta'$.

(2) implies (3). Let us suppose that $\gamma, \epsilon \prec \delta$ with $\gamma \neq \epsilon$. By Lemma 4.4 we can write $\delta \xrightarrow{u} \gamma$ and $\delta \xrightarrow{v} \epsilon$ for some $u, v \in L$ such that, by Lemma 3.4, $u \neq v$. Hence, u, v, δ_0 are pairwise distinct lower covers of δ_1.

We claim that, assuming (2), $\gamma_0 \wedge \epsilon_0 < u \wedge v$. Clearly, we have $\gamma_0 \wedge \epsilon_0 \leq u \wedge v$, so that, by the way of contradiction, let us assume that $\gamma_0 \wedge \epsilon_0 = u \wedge v$; it follows that $u \wedge v \leq \gamma_0$. Let $\delta' = (u, \delta_1)$ and let $\psi, \xi \in \mathbb{C}(L)$ be the covers determined by $\delta' \xrightarrow{\delta_0} \psi$ and $\delta' \xrightarrow{v} \xi$. It follows that $\xi_0 \leq \psi_0$, since $\xi_0 = u \wedge v \leq \gamma_0 = u \wedge \delta_0 = \psi_0$, thus contradicting property (2).

Next, we construct a lower bound β of γ, ϵ. Given that $\gamma_0 \wedge \epsilon_0 < u \wedge v$, we let $\beta_0 = \gamma_0 \wedge \epsilon_0$ and pick β_1 such that $\beta_0 \prec \beta_1 < u \wedge v$. Let us argue that $\beta = (\beta_0, \beta_1)$ is below γ. Clearly, $\beta_0 = \gamma_0 \wedge \epsilon_0 \leq \gamma_0$. Consequently $\beta_0 \leq \delta_0$ and, by construction, $\beta_1 \leq u \wedge v \leq \delta_1$. If $\beta_1 \leq \delta_0$ then $\beta_1 \leq \delta_0 \wedge u \wedge v = (\delta_0 \wedge u) \wedge (\delta_0 \wedge v) = \gamma_0 \wedge \epsilon_0 = \beta_0$, a contradiction; whence $\beta_1 \not\leq \delta_0$. We have argued that $\beta \leq \delta$ and $\beta_0 \leq \gamma_0$ which, together with $\gamma \leq \delta$, imply $\beta \leq \gamma$. In a similar way we show that $\beta \leq \epsilon$, so that β is a lower bound of γ, ϵ.

Finally, to see that β is the greatest lower bound of γ and ϵ, we argue as in Lemma 4.2. If $\alpha \leq \gamma, \epsilon$ then $\alpha_0 \leq \gamma_0 \wedge \epsilon_0 = \beta_0$ and, considering that $\alpha, \beta \leq \delta$ and L is pushdown, we deduce $\alpha \leq \beta$.

(3) implies (4). By Lemma 2.3.

(4) implies (5). Since $\mathbb{C}(L)$ has pullbacks, if $\mu \in \mathbb{C}(L)$ is a maximal cover, then the ideal it generates is a finite meet-semilattice and has a least element. By Lemma 4.1, it follows that L is join-semidistributive.

(5) implies (1). By Lemma 4.4. □

5 Lower Bounded Lattices

Recall that a lattice L is said to be *lower bounded* if there exists a finite set X and lattice epimorphism from the freely generated lattice $\mathcal{F}(X)$[3] onto L, say $f : \mathcal{F}(X) \twoheadrightarrow L$, such that, for each $y \in L$, the set $\{ x \mid y \leq f(x) \}$ is either empty or has a least element, see [14, §5] or [8, §2.13]. *Upper boundedness* is the dual notion of lower boundedness, and a lattice is said to be *bounded* if it is both lower and upper bounded.

There are already many characterizations of lower bounded lattices [6, 7, 20, 19] and this concept has also found applications within unexpected branches of lattice theory [10]. In this section we develop further the tools used in [4] to prove that lattices in $\mathcal{HH}$ – a class which generalizes the class of finite Coxeter lattices – are bounded. In this way we shall obtain a new characterization of lower bounded lattices. Our starting point will be the following classical result [8, §2.39]:

[3] If X contains at least three elements, then $\mathcal{F}(X)$ is an infinite lattice.

Theorem 5.1 (Johnsonn, Nation). *A lattice is lower bounded if and only if the* join-dependency relation *between join-irreducible elements contains no cycle.*

In order to use of this result, we must of course give the definition of the join-dependency relation D. Here we give its characterization in terms of the arrows relations [8, §11.10]. The reader can find in [8, §2.3] its standard definition.

Definition 5.2. For a lattice L and $j, k \in J(L)$, we let jDk if $j \neq k$ and there exists $m \in M(L)$ such that $j \uparrow m$ and $k \downarrow m$.

Let us introduce some more relations between join-irreducible elements of a lattice L:

Definition 5.3. For all $j, k \in J(L)$, we let:

- jAk if $j \neq k$ and, for some $m \in M(L)$, $j \uparrow m$ and $k \updownarrow m$,
- jBk if $j \neq k$ and, for some $m \in M(L)$, $j \updownarrow m$ and $k \downarrow m$,
- jCk if either jAk or jBk.

Let us remark that these relations are already known for semidistributive lattices [8, §2.5]. However, the definition presented here makes sense in any lattice. The next Lemma shows that, in the expression of Theorem 5.1, the D relation may be replaced by the C relation defined above.

Lemma 5.4. *The D relation has a cycle if and only if the C relation has a cycle. Consequently a lattice is lower bounded if and only if the C relation contains no cycle.*

Proof. Let us suppose that jDk and let $m \in M(L)$ be such that $j \uparrow m$ and $k \downarrow m$. Choose $l \in J(L)$ such that $l \updownarrow m$. If $l \in \{ j, k \}$, then jAk or jBk. If $l \neq j, k$, then jAl and lBk. Therefore one step of the relation D may be replaced by at most two steps of the relation C. Conversely, since $C \subseteq D$, every C-cycle gives rise to a D-cycle. $\square$

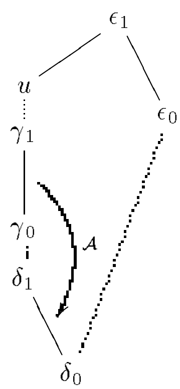
We shall introduce a number of relations and, to this goal, we first need the dual of the relation $\rightharpoondown$: for $w \in L$ and $\gamma, \delta \in \mathbb{C}(L)$, let us write $\gamma \underset{w}{\rightharpoonup} \delta$ if $w \neq \gamma_1$, $\delta_1 = w \vee \gamma_1$, and $\gamma_0 \prec w \leq \delta_0$. Also let us write $\gamma \rightharpoonup \delta$ if $\gamma \underset{w}{\rightharpoonup} \delta$ for some $w \in L$. The following remark is due now:

Lemma 5.5. *If a lattice L is join-semidistributive, then $\delta \rightharpoondown \gamma$ implies $\gamma \rightharpoonup \delta$. Consequently, if L is a semidistributive lattice, then $\delta \rightharpoondown \gamma$ holds if and only if $\gamma \rightharpoonup \delta$ holds.*

Proof. Let us suppose that $\delta \overset{u}{\rightharpoondown} \gamma$ so that, as usual, $\gamma_0 = u \wedge \delta_0 < \delta_0$. Thus, we can choose $w \in L$ such that $\gamma_0 \prec w \leq \delta_0$. Forcedly, $w \not\leq u$, since otherwise $\gamma_0 < w \leq u \wedge \delta_0 = \gamma_0$. It follows that $w \neq \gamma_1$ and that $w \vee u = \delta_1$. We claim that $w \vee \gamma_1 = \delta_1$. If this is not the case, then $w \vee \gamma_1 < \delta_1$ and the implication $(SD_\vee)$ fails: let $x = w \vee \gamma_1$, $y = u$, and $z = \delta_0$, then $x \vee y = w \vee \gamma_1 \vee u = w \vee u = \delta_1 = \gamma_1 \vee \delta_0 = w \vee \gamma_1 \vee \delta_0 = x \vee z$, but $x \vee (y \wedge z) = x \vee \gamma_0 = x < \delta_1 = x \vee y$. $\square$

Definition 5.6. For $\gamma, \delta \in \mathbb{C}(L)$, we let $\gamma \mathcal{A} \delta$ if there exists $\epsilon \in \mathbb{C}(L)$ and $u \in L$ such that $\epsilon \overset{u}{\rightharpoondown} \delta$ and $\delta_1 \leq \gamma_0 \prec \gamma_1 \leq u$. The dual relation is defined as follows: $\gamma \mathcal{B} \delta$ if there exists $\epsilon \in \mathbb{C}(L)$ and $u \in L$ such that $\epsilon \underset{u}{\rightharpoonup} \delta$ and $u \leq \gamma_0 \prec \gamma_1 \leq \delta_1$.

Intuitively, the $\mathcal{A}, \mathcal{B}$ relations express the dependency relation of covers in terms of the relations $\rightharpoondown$ and $\rightharpoonup$, when the configurations they give rise to form pentagons. The following picture illustrates this point with the $\mathcal{A}$ relation:



It would be tempting to define *facets* at this point. Intuitively, these are the configurations arising either from a relation of the form $\delta \overset{u}{\rightharpoondown} \gamma$, or from a relation of the form $\gamma \underset{w}{\rightharpoonup} \delta$. We believe, however, that the notion of facet is better suited to the semidistributive case, where the two relations $\rightharpoondown$ and $\rightharpoonup$ are one opposite of the other, as remarked in Lemma 5.5. Since we want to carry out our ideas in the weaker join-semidistributive setting, we delay the definition of facets until the end of this section when we shall come back to full semidistributivity.

The next two Lemmas exemplify the connections between the relations A and $\mathcal{A}$. The Lemmas apply to arbitrary lattices.

Lemma 5.7. *Let $j, k \in J(L)$ be such that, for some $\gamma, \delta \in \mathbb{C}(L)$, $\gamma \mathcal{A} \delta$, $(j_*, j) \leq \gamma$ and $(k_*, k) \leq \delta$. Then jAk.*

Proof. Let γ, δ be such that $\gamma \mathcal{A} \delta$ so that $\epsilon \overset{u}{\rightharpoondown} \delta$, for some $u \in L$ and $\epsilon \in \mathbb{C}(L)$, and moreover $\delta_1 \leq \gamma_0 \prec \gamma_1 \leq u$. Observe first that $j \neq k$, since $k \leq \delta_1 \leq \gamma_0$ and $j \not\leq \gamma_0$. Let μ be maximal in the set $\{\, \theta \in \mathbb{C}(L) \mid \epsilon \leq \theta \,\}$ so that μ is of the form $\mu = (m, m^*)$ for some $m \in M(L)$. We claim that $j \uparrow m$. We have on one side $j \leq \gamma_1 \leq u \leq \epsilon_1 \leq m^*$. Let us suppose that $j \leq m$: then $j \leq u \wedge \epsilon_1 \wedge m = u \wedge \epsilon_0 = \delta_0 \leq \gamma_0$. This is a contradiction, since $(j_*, j) \leq \gamma$ implies $j \not\leq \gamma_0$; hence $j \not\leq m$. We have shown that $j \uparrow m$; since $(k_*, k) \leq \delta \leq \epsilon \leq (m, m^*)$, we have $k \updownarrow m$, and therefore jAk. $\square$

In the next Lemma $\overset{*}{\rightharpoondown}$ denotes the reflexive transitive closure of the push-down relation $\rightharpoondown$. Since $\delta \overset{*}{\rightharpoondown} \gamma$ implies $\gamma \leq \delta$, the relation $\overset{*}{\rightharpoondown}$ is antisymmetric, and hence it is an ordering on $\mathbb{C}(L)$. We recall that this ordering coincides with the opposite of the order $\leq$ on $\mathbb{C}(L)$ if L is a pushdown lattice, see Proposition 3.5; on the other hand, the scope of the next Lemma is not restricted to pushdown lattices.

Lemma 5.8. *Let $j \in J(L)$ and $\delta \in \mathbb{C}(L)$ be such that $\delta_1 = j \vee \delta_0$. Then there exists $n \geq 0$ and a sequence (δ^i, γ^i), $i = 0, \ldots, n$, such that*

1. *$\gamma^0 = (j_*, j)$ and $\delta^n = \delta$,*
2. *$\delta_1^i = j \vee \delta_0^i$ and $\delta^i \xrightarrow{*} \gamma^i$, for $i = 0, \ldots, n$,*
3. *$\delta^{i-1} \mathcal{A} \gamma^i$, for $i = 1, \ldots, n$.*

Proof. The proof is by induction on the height of the interval $[j, \delta_1]$.

Consider the set

$$PD(\delta, j) = \{\, \theta \in \mathbb{C}(L) \mid \delta \xrightarrow{*} \theta \text{ and } j \vee \theta_0 = \theta_1 \,\},$$

observe that $\delta \in PD(\delta, j)$, and hence pick $\mu \in PD(\delta, j)$ which is maximal w.r.t. the relation $\xrightarrow{*}$.

If $j = \mu_1$, then μ_1 is join-irreducible so that $\mu_0 = j_*$ and the statement holds with $n = 0$, $\delta^0 = \delta$, and $\gamma^0 = \mu = (j_*, j)$.

If $j \neq \mu_1$, then $j < \mu_1$ and we can find $u \in L$ such that $j \leq u \prec \mu_1$. As usual, observe that $u \neq \mu_0$, since $j \not\leq \mu_0$. We let therefore $\gamma_0 = u \wedge \mu_0$ and $\delta_1' = j \vee \gamma_0$. We observe next two facts about the pair (γ_0, δ_1'): (a) we have $\gamma_0 < \delta_1'$, since if $\gamma_0 = \delta_1' = j \vee \gamma_0$, then $j \leq \gamma_0 \leq \mu_0$, contradicting $j \vee \mu_0 = \mu_1$; (b) the pair (γ_0, δ_1') is not a cover, since otherwise, defining $\delta' = (\gamma_0, \delta_1')$, then $\mu \xrightarrow{u} \delta'$ and $\delta_1' = j \vee \gamma_0$ imply that μ is not maximal in the set $PD(\delta, j)$. As a consequence of our observations, we can find $\gamma_1, \delta_0' \in L$ such that $\gamma_0 \prec \gamma_1 \leq \delta_0' \prec \delta_1' \leq u$. If we let $\gamma = (\gamma_0, \gamma_1)$ and $\delta' = (\delta_0', \delta_1')$, then $\mu \xrightarrow{u} \gamma$ and $\delta' \mathcal{A} \gamma$; consequently, $\delta \xrightarrow{*} \gamma$ as well.

Next we remark that $[j, \delta_1'] \subset [j, \delta_1]$, since $\delta_1' \leq u < \delta_1$. We can use the inductive hypothesis to find $m \geq 0$ and a sequence (δ^i, γ^i), $i = 0, \ldots m$, satisfying *(2)* and *(3)* and such that $\gamma^0 = (j_*, j)$ and $\delta^m = \delta'$. Therefore we let $n = m + 1$ and, to obtain a sequence satisfying *(1)*, *(2)*, and *(3)*, we append the pair (δ, γ) to the sequence (γ^i, δ^i), $i = 0, \ldots, m$. □

Following [4], we are ready to introduce strict facet labellings.

Definition 5.9. A *strict lower facet labelling* of a lattice L is a function $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ such that $f(\gamma) = f(\delta)$ if $\delta \rightharpoondown \gamma$ and $f(\gamma) < f(\delta)$ if $\delta \mathcal{A} \gamma$. A *strict upper facet labelling* is defined dually: it is a function $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ such that $f(\gamma) = f(\delta)$ if $\delta \rightharpoonup \gamma$ and $f(\gamma) < f(\delta)$ if $\delta \mathcal{B} \gamma$.

A *strict facet labelling* of a lattice L is a function $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ which is both a strict lower facet labelling and a strict upper facet labelling.

Lemma 5.10. *If L is a pushdown lattice, f is a strict lower facet labelling, and $\gamma \leq \delta$, then $f(\gamma) = f(\delta)$.*

Proof. If $\gamma \leq \delta$ then by Lemma 4.4 we can find a path of the relation $\rightharpoondown$ from δ to γ. The statement follows since f is constant on the relation $\rightharpoondown$. $\square$

Lemma 5.11. *Let L be a lattice with a strict lower facet labelling f. Let $j \in J(L)$ and $\delta \in \mathbb{C}(L)$ be such that $\delta_1 = j \vee \delta_0$. Then $f(\delta) \leq f(j_*, j)$ and $f(\delta) = f(j_*, j)$ implies $(j_*, j) \leq \delta$.*

Proof. By Lemma 5.8, let $n \geq 0$ and (δ^i, γ^i), $i = 0, \ldots, n$, be such that $\delta^i \overset{*}{\rightharpoondown} \gamma^i$ for $i = 0, \ldots, n$, $\delta^{i-1} \mathcal{A} \gamma^i$ for $i = 1, \ldots, n$, $\delta^n = \delta$ and $\gamma^0 = (j, j_*)$.

Since f is constant on $\rightharpoondown$, it is constant on its reflexive transitive closure $\overset{*}{\rightharpoondown}$. Hence, we have $f(\delta^i) = f(\gamma^i)$ for $i = 0, \ldots, n$, and $f(\gamma^i) < f(\delta^{i-1})$ for $i = 1, \ldots, n$. We deduce that $f(\delta) = f(\delta^n) \leq f(\gamma^0) = f(j_*, j)$ and, $f(\delta) < f(j_*, j)$ if $n \geq 1$. Therefore, if $f(\delta) = f(j_*, j)$, then $n = 0$, $\delta \overset{*}{\rightharpoondown} (j_*, j)$ and $(j_*, j) \leq \delta$. $\square$

By duality, we obtain the following Corollary.

Corollary 5.12. *Let L be a lattice with an upper strict facet labelling f. Let $m \in M(L)$ and $\delta \in \mathbb{C}(L)$ such that $m \wedge \delta_1 = \delta_0$. Then $f(\delta) \leq f(m, m^*)$ and $f(\delta) = f(m, m^*)$ implies $\delta \leq (m, m^*)$.*

We are ready to achieve the first goal of this section, Proposition 5.15.

Lemma 5.13. *If L is a join-semidistributive lattice with a strict lower facet labelling f, then jAk implies $f(k_*, k) < f(j_*, j)$.*

Proof. If jAk then $j \neq k$, $j \uparrow m$, and $(k_*, k) \leq (m, m^*)$ for some $m \in M(L)$. It follows from Lemma 5.10 that $f(k_*, k) = f(m, m^*)$. Since moreover we have $j \vee m = m^*$, Lemma 5.11 implies that $f(m, m^*) \leq f(j_*, j)$ and moreover that this is an inequality: otherwise, if $f(m, m^*) = f(j_*, j)$, then $(j_*, j) \leq (m, m^*)$; consequently, $j = k$ since in a join-semidistributive lattice there exists at most one $j \in \mathbb{C}(L)$ such that $j \updownarrow m$. $\square$

The next Lemma is not a mere consequence of duality.

Lemma 5.14. *If L is a join-semidistributive lattice with a strict upper facet labelling f, then jBk implies $f(k_*, k) < f(j_*, j)$.*

Proof. Let $j, k \in J(L)$ be such that jBk, that is, $j \neq k$ and, for some $m \in M(L)$, $j \updownarrow m$ and $k \downarrow m$. Since by Lemma 3.3 L is a pushup lattice, we can make use of Lemma 5.10 to deduce that $f(j_*, j) = f(m, m^*)$. On the other hand, Lemma 5.12 ensures that $f(k_*, k) \leq f(m, m^*)$ and $(k_*, k) \leq (m, m^*)$ if $f(k_*, k) = f(m, m^*)$. We cannot have $f(k_*, k) = f(m, m^*)$, since then $(k_*, k) \leq (m, m^*)$, $k \updownarrow m$, and $j = k$ by join-semidistributivity. Hence $f(k_*, k) < f(m, m^*) = f(j_*, j)$. $\square$

Proposition 5.15. *A join-semidistributive lattice with a strict facet labelling is lower bounded.*

Proof. According to Lemma 5.4, it is enough to show that the relation C has no cycle, for which we shall argue that jCk implies $f(k_*, k) < f(j_*, j)$. If jAk then we use Lemma 5.13. If jBk then we can use Lemma 5.14. $\square$

Our next goal is to prove the converse of Proposition 5.15.

Lemma 5.16. *Let L be a join-semidistributive lattice. If $j, k \in J(L)$ are such that for some $\gamma, \delta \in \mathbb{C}(L)$, $\gamma \mathcal{B} \delta$, $(j_*, j) \leq \gamma$ and $(k_*, k) \leq \delta$, then jBk.*

Proof. Let γ, δ be such that $\gamma \mathcal{B} \delta$: for some $u \in L$ and $\epsilon \in \mathbb{C}(L)$, $\epsilon \xrightarrow[u]{} \delta$ and $\epsilon_0 \prec u \leq \gamma_0 \prec \gamma_1 \leq \delta_0$. Let j, k be as in the statement, and observe that $j \neq k$, since $j \leq \gamma_1 \leq \delta_0$ and $k \not\leq \delta_0$. Let $\mu = (m, m^*)$ be maximal above γ, so that $j \updownarrow m$, and let $l \in J(L)$ such $(l_*, l) \leq \epsilon$. We have $m \geq \gamma_0 \geq u \geq \epsilon_0 \geq l_*$. If $m \geq l$, then $m \geq u \vee \epsilon_0 \vee l = u \vee \epsilon_1 = \delta_1$; this is in turn implies that $m \geq \delta_1 \geq \gamma_1$, a contradiction. Hence $m \not\geq l$ and $l \downarrow m$. We have therefore jBl. It is now easy to see that $l = k$, since $(l_*, l) \leq \delta$ and $(k_*, k) \leq \delta$ imply $l = k$, by join-semidistributivity. $\square$

Proposition 5.17. *A lower bounded lattice has a strict facet labelling.*

Proof. If L is bounded then it is join-semidistributive [8, §2.20] and the join-dependency relation D is acyclic [8, §2.39]. Let us denote by $\trianglelefteq$ its reflexive and transitive closure, and let $g : J(L) \longrightarrow \mathbb{N}$ be an antilinear extension of the poset $\langle J(L), \trianglelefteq \rangle$. That is, g is such that $g(y) < g(x)$ whenever xDy.

We define $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ as follows: $f(\delta) = g(j(\delta))$ where $j(\delta) \in J(L)$ is the unique join-irreducible element such that $(j(\delta)_*, j(\delta)) \leq \delta$. Then f is a strict facet labelling. It is a lower facet labelling: if $\delta \rightharpoondown \gamma$ then $j(\delta) = j(\gamma)$ and $f(\delta) = g(j(\delta)) = g(j(\gamma)) = f(\gamma)$; if $\delta \mathcal{A} \gamma$, then $j(\delta) A j(\gamma)$, by Lemma 5.7, and $j(\delta) D j(\gamma)$; it follows that $f(\gamma) = g(j(\gamma)) < g(j(\delta)) = f(\delta)$. Similarly, it is an upper strict facet labelling, since $\delta \xrightarrow[u]{} \gamma$ implies $j(\gamma) = j(\delta)$ and $\delta \mathcal{B} \gamma$ implies $j(\delta) B j(\gamma)$, by Lemma 5.16, and therefore $f(\gamma) < f(\delta)$. $\square$

Recalling that lower bounded lattices are join-semidistributive [8, §2.20], we end this section collecting the observations presented so far into a main result:

Theorem 5.18. *A lattice is lower bounded if and only if it is join-semidistributive and has a strict facet labelling.*

Finally, we observe that from Theorem 5.18 it is quite immediate to derive a standard result by Day, see [8, §2.64], stating that a semidistributive lower bounded lattice is bounded. Recall now that in a semidistributive lattice $\delta \rightharpoondown \gamma$ holds if and only if $\gamma \rightharpoonup \delta$ holds. Theorem (5.18) leads to a characterization of bounded lattices, that we shall rephrase in a language closer to [4]. Call a *facet* of L a quadruple of distinct covers $(\delta, \delta', \gamma, \gamma') \in \mathbb{C}^4(L)$ such that

$$\delta_1 = \delta_1' \text{ and } \gamma_0 = \gamma_0' \text{, and moreover } \delta \xrightarrow{\delta_0'} \gamma \text{ and } \delta' \xrightarrow{\delta_0} \gamma' \text{.}$$

See Figure 1 in the Introduction. Say that ϵ is *interior* to the facet $(\delta, \delta', \gamma, \gamma')$ if $\gamma_1 \leq \epsilon_0 \prec \epsilon_1 \leq \delta_0'$ or $\gamma_1' \leq \epsilon_0 \prec \epsilon_1 \leq \delta_0$.

Theorem 5.19. *A lattice is bounded if and only if it is semidistributive and there exists a function $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ such that for each facet $(\delta, \delta', \gamma, \gamma')$ $f(\delta) = f(\gamma)$ and $f(\delta') = f(\gamma')$, and moreover $f(\delta'), f(\gamma) < f(\epsilon)$ whenever ϵ is interior to such a facet.*

6 Derived Semidistributive Lattices

The main result of this section is Theorem 6.5 stating that a poset of the form $\mathbb{C}(L,\gamma)$, $\gamma \in \mathbb{C}(L)$, is a semidistributive lattice whenever L is semidistributive. Observe that if $\gamma = (j_*, j)$ with $j \in J(L)$ and L is join-semidistributive, then $\mathbb{C}(L,\gamma)$ is the set $\{\, \delta \in \mathbb{C}(L) \mid \gamma \leq \delta \,\}$. With this in mind we observe:

Proposition 6.1. *If L is a semidistributive lattice, then $\mathbb{C}(L,\gamma)$ is a lattice, for each $\gamma \in \mathbb{C}(L)$.*

Proof. By Proposition 4.5 and its dual $\mathbb{C}(L)$ has pullbacks and pushouts, hence $\mathbb{C}(L,\gamma)$ is a lattice by Corollary 2.6. $\square$

We shall call $\mathbb{C}(L,\gamma)$ the semidistributive lattice *derived* from L and γ. We study next additional properties of the lattices of the form $\mathbb{C}(L,\gamma)$. The next Lemma will prove useful in establishing these properties.

Lemma 6.2. *Let L be a join-semidistributive lattice, and let $\gamma, \delta, \epsilon \in \mathbb{C}(L)$ be such that $\gamma \leq \delta$, $\epsilon \prec \delta$, and $\gamma \not\leq \epsilon$. Then $\gamma_0 \vee \epsilon_1 = \delta_1$.*

Proof. From $\gamma \leq \delta$ and $\epsilon \prec \delta$, it follows that $\gamma_1 \vee \epsilon_1 \leq \delta_1$. We claim that $\gamma_1 \vee \epsilon_1 = \delta_1$. Observe that if the claim holds, then we can easily derive the conclusion of the Lemma, $\gamma_0 \vee \epsilon_1 = \delta_1$, since by join-semidistributivity $\gamma_1 \vee \epsilon_1 = \delta_1 = \delta_0 \vee \epsilon_1$ implies $\gamma_0 \vee \epsilon_1 = (\gamma_1 \wedge \delta_0) \vee \epsilon_1 = \gamma_1 \vee \epsilon_1 = \delta_1$.

Let us suppose next that the claim does not hold, i.e. that $\gamma_1 \vee \epsilon_1 < \delta_1$. Choose $u \in L$ such that $\gamma_1 \vee \epsilon_1 \leq u \prec \delta_1$ and observe that $u \neq \delta_0$, since otherwise $\gamma, \epsilon \not\leq \delta$. Let $\epsilon' \in \mathbb{C}(L)$ be such that $\delta \xrightarrow{u} \epsilon'$. Since $\gamma_0 \leq \gamma_1 \leq u$ and $\epsilon_0 \leq \epsilon_1 \leq u$, Proposition 3.6 implies that $\gamma \leq \epsilon'$ and $\epsilon \leq \epsilon'$. In turn, $\epsilon \leq \epsilon' < \delta$ and $\epsilon \prec \delta$ imply $\epsilon = \epsilon'$. We have therefore $\gamma \leq \epsilon' = \epsilon$, contradicting the hypothesis of the Lemma, $\gamma \not\leq \epsilon$. $\square$

In the following, we shall use capital Greek letters to range on elements of $\mathbb{C}(\mathbb{C}(L,\gamma))$. Observe that the next Propositions make sense: if L is join-semidistributive then by Proposition 4.5 $\mathbb{C}(L,\gamma)$ is a finite poset with pullbacks. This is enough to ensure that the relation $\leq$ on $\mathbb{C}(\mathbb{C}(L,\gamma))$ is a transitive relation. Since this relation is clearly reflexive and antisymmetric, it is a partial ordering.

Proposition 6.3. *If L if a join-semidistributive lattice then the projection*

$$(\,\cdot\,)_0 : \mathbb{C}(\mathbb{C}(L,\gamma)) \longrightarrow \mathbb{C}(L,\gamma)$$

is a Grothendieck fibration.

Proof. Let $\Gamma, \Psi, \Delta \in \mathbb{C}(\mathbb{C}(L,\gamma))$ be such that $\Gamma, \Psi \leq \Delta$ and $\Gamma_0 \leq \Psi_0$. We shall prove that $\Gamma_1 \leq \Psi_1$, which in turn implies that $\Gamma \leq \Psi$.

Observe that, since $\Gamma_1, \Psi_1 \leq \Delta_1$, the pullback $\Gamma_1 \wedge \Psi_1$ exists in $\mathbb{C}(L)$. Moreover, since $\Gamma_0 \leq \Psi_0 \leq \Psi_1$, we have $\Gamma_0 \leq \Gamma_1 \wedge \Psi_1 \leq \Gamma_1$ and therefore either $\Gamma_1 \wedge \Psi_1 = \Gamma_1$, or $\Gamma_1 \wedge \Psi_1 = \Gamma_0$, since $\Gamma_0 \prec \Gamma_1$. We shall show that $\Gamma_1 \wedge \Psi_1 = \Gamma_0$ cannot occur, hence we have $\Gamma_1 \wedge \Psi_1 = \Gamma_1$, that is, $\Gamma_1 \leq \Psi_1$.

By the way of contradiction, assume that $\Gamma_1 \wedge \Psi_1 = \Gamma_0$. Recall from Theorem 4.3 that $\Gamma_{1,0} \wedge \Psi_{1,0} = (\Gamma_1 \wedge \Psi_1)_0$, and hence we see that $\Gamma_{1,0} \wedge \Psi_{1,0} = (\Gamma_1 \wedge \Psi_1)_0 = \Gamma_{0,0} \leq \Delta_{0,1}$.

Next, we make use of Lemma 6.2 to prove that $\Gamma_{1,0} \wedge \Psi_{1,0} \not\leq \Delta_{0,1}$, thus obtaining a contradiction. By definition of the poset $\mathbb{C}(\mathbb{C}(L))$, $\Delta_0 \prec \Delta_1$, $\Gamma_1 \leq \Delta_1$, $\Gamma_1 \not\leq \Delta_0$ and, similarly, $\Psi_1 \leq \Delta_1$ but $\Psi_1 \not\leq \Delta_0$. In the statement of Lemma 6.2 let $\gamma = \Gamma_1$, $\delta = \Delta_1$, $\epsilon = \Delta_0$, and deduce $\Gamma_{1,0} \vee \Delta_{0,1} = \Delta_{1,1}$. Similarly, let $\gamma = \Psi_1$, $\delta = \Delta_1$, $\epsilon = \Delta_0$, and deduce $\Psi_{1,0} \vee \Delta_{0,1} = \Delta_{1,1}$. Therefore, from $\Gamma_{1,0} \vee \Delta_{0,1} = \Delta_{1,1} = \Psi_{1,0} \vee \Delta_{0,1}$ we deduce $(\Gamma_{1,0} \wedge \Psi_{1,0}) \vee \Delta_{0,1} = \Delta_{1,1}$ and, consequently, $\Gamma_{1,0} \wedge \Psi_{1,0} \not\leq \Delta_{0,1}$. □

Proposition 6.4. *If L is join-semidistributive, then the projection $(\,\cdot\,)_0$ creates pullbacks.*

Proof. Since the projection $(\,\cdot\,)_0$ is a conservative Grothendieck fibration, it is enough by Lemma 4.2 to prove that if $\Gamma, \Psi \leq \Delta$, then there exists $\Upsilon \leq \Gamma, \Psi$ such that $\Upsilon_0 = \Gamma_0 \wedge \Psi_0$.

To this goal, we observe first that $\Gamma_0 \wedge \Psi_0 < \Gamma_1 \wedge \Psi_1$: as in proof the previous Proposition, we have $(\Gamma_1 \wedge \Psi_1)_0 \vee \Delta_{0,1} = \Delta_{1,1}$, $(\Gamma_1 \wedge \Psi_1)_0 \not\leq \Delta_{0,1}$, and consequently $\Gamma_1 \wedge \Psi_1 \not\leq \Delta_0$. Hence the standard relation $\Gamma_0 \wedge \Psi_0 \leq \Gamma_1 \wedge \Psi_1$ is not an equality, since $\Gamma_0 \wedge \Psi_0 \leq \Delta_0$, and $\Gamma_0 \wedge \Psi_0 < \Gamma_1 \wedge \Psi_1$.

Let $(\Upsilon_0, \Upsilon_1) \in \mathbb{C}^2(L)$ be such that $\Gamma_0 \wedge \Psi_0 = \Upsilon_0 \prec \Upsilon_1 \leq \Gamma_1 \wedge \Psi_1$ and observe that the following are equivalent: $\Upsilon_1 \leq \Gamma_0$, $\Upsilon_1 \leq \Delta_0$, $\Upsilon_1 \leq \Psi_0$, $\Upsilon_1 \leq \Gamma_0 \wedge \Psi_0 = \Upsilon_0$. For example, if $\Upsilon_1 \leq \Gamma_0$, then $\Upsilon_1 \leq \Gamma_0 \leq \Delta_0$, and since $\Upsilon_1 \leq \Psi_1$, then $\Upsilon_1 \leq \Delta_0 \wedge \Psi_1 = \Psi_0$ and $\Upsilon_1 \leq \Gamma_0 \wedge \Upsilon_0 = \Psi_0$.

Since $\Upsilon_1 \not\leq \Upsilon_0$, we deduce $\Upsilon_1 \not\leq \Gamma_0$ and $\Upsilon_1 \not\leq \Psi_0$. Therefore we have all the elements to state that $\Upsilon \leq \Gamma, \Delta$ and, by construction, we have $\Upsilon_0 = \Gamma_0 \wedge \Delta_0$. □

Using Proposition 6.4, also in its dual form, we arrive to the first achievement of this section.

Theorem 6.5. *If L is a semidistributive lattice, then $\mathbb{C}(L, \gamma)$ is a semidistributive lattice, for each $\gamma \in \mathbb{C}(L)$.*

We shall use next the characterization of bounded lattices of Theorem 5.19 to obtain the second main result of this section.

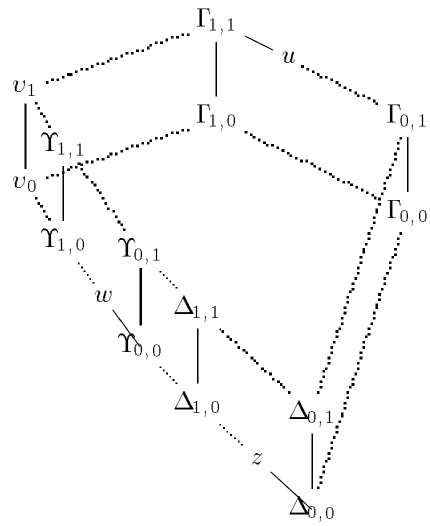
Theorem 6.6. *If L is a bounded lattice, then so is $\mathbb{C}(L, \gamma)$, for each $\gamma \in \mathbb{C}(L)$.*

Proof. Since L is bounded, then Theorem 5.19 ensures that it has a strict facet labelling $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$. Recall from Lemma 4.4 that a cover in $\mathbb{C}(L)$ is of the form (Γ_0, Γ_1) with $\Gamma_1 \overset{u}{\rightharpoondown} \Gamma_0$ for a unique lower cover u of $\Gamma_{1,1}$. Therefore we define a function $F : \mathbb{C}(\mathbb{C}(L, \gamma)) \longrightarrow \mathbb{N}$ by

$$F(\Gamma) = f(u, \Gamma_{1,1})\,.$$

Observe that if $\Gamma_1 \overset{u}{\rightharpoondown} \Gamma_0$ and $\Gamma_0 \underset{w}{\rightharpoondown} \Gamma_1$, then $(u, \Gamma_{1,1}) \overset{\Gamma_{1,0}}{\rightharpoondown} (\Gamma_{0,0}, w)$ so that $f(\Gamma_{0,0}, w) = f(u, \Gamma_{1,1}) = F(\Gamma)$. Therefore the definition of F does not depend on whether we choose the relation $\rightharpoondown$ or its dual $\rightharpoonup$.

Next we prove that F so defined is a strict facet labelling. To this goal, let us suppose that $\Gamma \xrightarrow{v} \Delta$ and $\Gamma_1 \leq \Upsilon_0 \prec \Upsilon_1 \leq v$, as sketched in the next diagram:



Let us also suppose that $\Gamma_1 \xrightarrow{u} \Gamma_0$, $\Delta_0 \xrightarrow[z]{} \Delta_1$, and $\Upsilon_0 \xrightarrow[w]{} \Upsilon_1$, so that $F(\Gamma) = f(u, \Gamma_{1,1})$, $F(\Delta) = f(\Delta_{0,0}, z)$, and $F(\Upsilon) = F(\Upsilon_{0,0}, w)$.

Recall that $\Delta_0 = v \wedge \Gamma_0$ and therefore $\Delta_{0,0} = v_0 \wedge \Gamma_{0,0} = v_0 \wedge \Gamma_{1,0} \wedge u = v_0 \wedge u$. Since $\Delta_0 \prec z \leq v_0$, then $z \not\leq u$, otherwise $z \leq v_0 \wedge u = \Delta_{0,0}$. We have therefore $z \leq \Gamma_{1,1}$, $\Delta_{0,0} \leq u$, $z \not\leq u$, that is $(\Delta_{0,0}, z) \leq (u, \Gamma_{1,1})$. Consequently, $F(\Delta) = f(\Delta_{0,0}, z) = f(u, \Gamma_{1,1}) = F(\Gamma)$.

In order to show that $F(\Delta) < F(\Upsilon)$ it is enough to show that if $j, k \in J(L)$, $(j_*, j) \leq (\Upsilon_{0,0}, w)$, and $(k_*, k) \leq (u, \Gamma_{1,1})$, then jAk. It follows then, by Lemma 5.13, that

$$F(\Delta) = F(\Gamma) = f(u, \Gamma_{1,1}) = f(k_*, k) < f(j_*, j) \leq f(\Upsilon_0, w) = F(\Upsilon).$$

Let $m \in M(L)$ such that $(u, \Gamma_{1,1}) \leq (m, m^*)$. We have $j \leq w \leq \Gamma_{1,1} \leq m^*$, and if $j \leq m$, then $j \leq v_0 \wedge \Gamma_{1,1} \wedge m = v_0 \wedge u = \Delta_{0,0} \leq \Upsilon_{0,0}$, a contradiction.

By duality, it also follows that $F(v, \Gamma_1) < F(\Upsilon)$. $\square$

7 Derived Lattices of Newman Lattices

We refer the reader to [2, 3, 5, 18] for introductory readings on Newman lattices. In this section we explicitly compute derived lattices $\mathbb{C}(L, \alpha)$ when L is a Permutohedron or an Associahedron and $\alpha \in \mathbb{C}(L)$ is an *atomic cover* of L, i.e. it is of the form $\alpha = (\perp, \alpha_1)$ with α_1 an atom of L. We shall see that these derived lattices are again Permutohedra (respectively, Associahedra) of same dimension minus one. We remark therefore a peculiar property of these lattices, they are *regular*, meaning that, up to isomorphism, the shape of $\mathbb{C}(L, \alpha)$ does not depend on the choice of the atomic cover α. We shall exhibit later a semidistributive lattice – not complemented – that it not regular. Regularity is a reminiscent property of Boolean algebras: if $\mathcal{B}^n$ is the Boolean algebra with n atoms, α_1 being one of them, then the equality $\mathbb{C}(\mathcal{B}^n, \alpha) = \mathcal{B}^{n-1}$ holds up to isomorphism. More generally:

Proposition 7.1. *If α is an atomic cover of a distributive lattice L, then the projection $(\,\cdot\,)_0$ from $\mathbb{C}(L,\alpha)$ to the lower set $\{\, x \in L \mid \alpha_1 \not\leq x \,\}$ is an isomorphism.*

The Proposition depends on modularity, since if $\alpha_1 \not\leq x$, then $x \prec x \vee \alpha_1$.

In the following proofs we shall intensively use the category of finite ordinals and functions among them, a skeleton of the category of finite sets and functions. To this goal, let $[n]$ be the set $\{\, 1, \ldots, n \,\}$ and, for $i \in [n]$, denote by $\hat{i}_n : [n-1] \longrightarrow [n]$ the unique order preserving injection whose image is $[n] \setminus \{\, i \,\}$. For $k \in [n-1]$ denote by $N_n^k : [n] \longrightarrow [n-1]$ the unique order preserving surjection such that $N_n^k(k) = N_n^k(k+1)$. As the subscripts n will always be understood from the context, we shall omit them and write only $\hat{i}$ and N^k.

Proposition 7.2. *Let $\mathcal{S}_n$ be the Permutohedron on n letters (i.e. the weak Bruhat order on permutations on n elements). If α is an atomic cover of $\mathcal{S}_n$ then $\mathbb{C}(\mathcal{S}_n, \alpha)$ is isomorphic to $\mathcal{S}_{n-1}$.*

Proof. As usual we represent a permutation $w \in \mathcal{S}_n$ as the word $w(1)\ldots w(n) = w_1 \ldots w_n$. An *increase* of w is an index $i \in \{\, 1, \ldots, n-1 \,\}$ such that $w_i < w_{i+1}$. If i is an increase of w and σ^i denotes the exchange permutation $(i, i+1)$, then we represent the cover $w \prec w \circ \sigma^i$ of $\mathcal{S}_n$ by the pair (w, i). Every cover arises in this way.

Remark next that a cover (w, i) is perspective to the atomic cover $(\bot, \sigma^k)$ if and only if $w_i = k$ and $w_{i+1} = k+1$. If (w, i) is such a cover, then we define $\psi^k(w, i)$ as the compose

$$\begin{tikzcd}
{[n]} \arrow[r, "w"] & {[n]} \arrow[d, "N^k"] \\
{[n-1]} \arrow[u, "\hat{i}"] \arrow[r, "\psi^k(w,i)"] & {[n-1]}
\end{tikzcd}$$

For example, $\psi^2(45231, 3)$ is computed as follows. We first erase the letter in third position and obtain the word 4531. Then we normalize this word to a permutation. To this goal, knowing that in third position of the original word there was the letter 2, we must decrease by one all the values of this word that are strictly greater than 2. Thus, we obtain the permutation 3421. We remark that we would have obtained the same result if we first erase the letter after the third position (i.e. in forth position) and then decrease by one all the values greater than 3. More generally, we could have equivalently defined $\psi^k(w, i)$ as the compose $N^{k+1} \circ w \circ \widehat{i+1}$.

The informal example already suggests that $\psi^k(w, i)$ is injective, and hence it is bijective; let us argue formally in this sense. If $\psi^k(w, i)$ is not injective, then there exists $x, y \in [n]$ such that x, y, i are pairwise distinct and $N^k(w_x) = N^k(w_y)$. But this may happen only if $\{ w_x, w_y \} = \{ k, k+1 \}$ and, by the assumption on the cover (w, i) stating that $w_i = k$ and $w_{i+1} = k+1$, this happens exactly when $\{ x, y \} = \{ i, i+1 \}$.

It is easily seen that ψ^k is a bijection from $\mathbb{C}(\mathcal{S}_n, (\perp, \sigma^k))$ to $\mathcal{S}_{n-1}$: if $u \in \mathcal{S}_{n-1}$, then there exists a unique cover (w, i) of $\mathcal{S}_n$ which is sent by ψ^k to u. The position $i \in [n-1]$ is the unique index i such that $u_i = k$ and then we define w as follows:

$$w_j = \begin{cases} k\,, & j = i\,, \\ \hat{k}(u_{N^i(j)})\,, & \text{otherwise}\,. \end{cases}$$

To prove that ψ^k is an order isomorphism, we prove that $(w, i) \prec (w', i')$ if and only if $\psi^k(w, i) \prec \psi^k(w', i')$. This equivalence is an immediate consequence of the following two claims.

Claim 7.3: For $j \in [n-2]$, j is an increase of $\psi^k(w, i)$ if and only if $\hat{i}(j)$ is an increase of w.

Since $w_i = k$, then $w_{\hat{i}(j)} \neq k$ for each $j \in [n-1]$, so that $\hat{k}$ and N^k are inverse order preserving bijections relating the sets $[n-1]$ and $\{ w_{\hat{i}(j)} \mid j \in [n-1] \} = [n] \setminus \{ k \}$. Thus we have

$$\begin{aligned} \psi^k(w, i)_j < \psi^k(w, i)_{j+1} &\text{ iff } N^k(w_{\hat{i}(j)}) < N^k(w_{\hat{i}(j+1)}) \\ &\text{ iff } w_{\hat{i}(j)} < w_{\hat{i}(j+1)}\,, \end{aligned}$$

and we are left to argue that

$$w_{\hat{i}(j)} < w_{\hat{i}(j+1)} \text{ iff } w_{\hat{i}(j)} < w_{\hat{i}(j)+1}\,.$$

Clearly this is the case if $\hat{i}(j) + 1 = \hat{i}(j+1)$. If not, then $\hat{i}(j) = i - 1$, and we need to argue that $w_{i-1} < w_i$ if and only if $w_{i-1} < w_{i+1}$. This is an immediate consequence of the assumptions w, namely that $w_i = k < k+1 = w_{i+1}$. □ *Claim*

The previous claim implies that the following data are in bijection: the upper covers of $\psi^k(w, i)$, the increases of $\psi^k(w, i)$, the increases of w that are distinct of i, the upper covers of w that are distinct from $w \circ \sigma^i$, the upper covers of (w, i) in the poset $\mathbb{C}(\mathcal{S}_n)$. The next claim shows that ψ^k respects this bijection.

Claim 7.4: Given an increase j of $\psi^k(w, i)$, let w' and i' be determined by the pushup relation $(w, i) \xrightarrow[w \circ \sigma^{\hat{i}(j)}]{} (w', i')$. Then

$$\psi^k(w', i') = \psi^k(w, i) \circ \sigma^j\,.$$

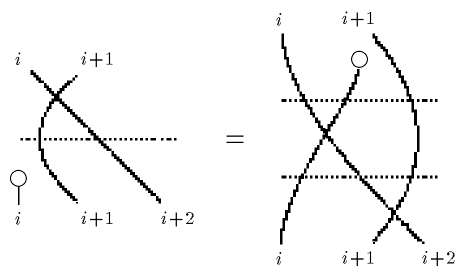
Firstly, we claim first that the following relations hold:

$$\hat{i} \circ \sigma^j = \sigma^{\hat{i}(j)} \circ \hat{i}\,, \qquad \text{if } |\hat{i}(j) - i| > 1\,, \tag{2}$$

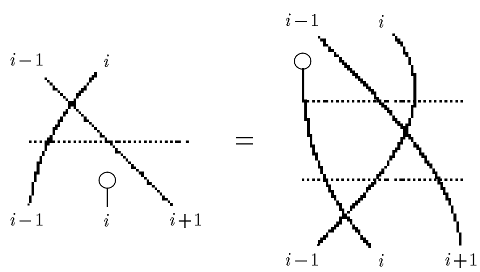
$$\hat{i} \circ \sigma^j = \sigma^{\hat{i}(j)} \circ \sigma^i \circ \widehat{\hat{i}(j)}\,, \qquad \text{otherwise, if } |\hat{i}(j) - i| = 1\,. \tag{3}$$

We leave the reader to verify equation (2) and focus instead on (3). We shall split its verification into two obvious cases and suggest a proof by making use of standard strings diagrams, see for example [13, §2.3].

Case $\hat{i}(j) = i+1$, that is $j = i$. Equation (3) reduces to $\hat{i} \circ \sigma^i = \sigma^{i+1} \circ \sigma^i \circ \widehat{i+1}$, which is easily seen to hold by comparing the following two diagrams:



Case $\hat{i}(j) = i-1$, that is $j = i-1$. Equation (3) reduces to $\hat{i} \circ \sigma^{i-1} = \sigma^{i-1} \circ \sigma^i \circ \widehat{i-1}$, which is easily seen to hold by comparing the following two diagrams:



Next, we compute w' and i' in the pushup $(w, i) \xrightarrow[w \circ \sigma^{\hat{i}(j)}]{} (w', i')$. If $|\hat{i}(j) - i| > 1$, then $w' = w \circ \sigma^{\hat{i}(j)}$ and $i' = i$. Otherwise, if $|\hat{i}(j) - i| = 1$, then $w' = w \circ \sigma^{\hat{i}(j)} \circ \sigma^i$ and $i' = \hat{i}(j)$.

Therefore, if $|\hat{i}(j) - i| > 1$, then

$$\begin{aligned} \psi^k(w', i') &= \psi^k(w \circ \sigma^{\hat{i}(j)}, i) \\ &= N^k \circ w \circ \sigma^{\hat{i}(j)} \circ \hat{i} \\ &= N^k \circ w \circ \hat{i} \circ \sigma^j = \psi^k(w, i) \circ \sigma^j \,, \end{aligned}$$

and, if $|\hat{i}(j) - i| = 1$, then

$$\begin{aligned} \psi^k(w', i') &= \psi^k(w \circ \sigma^{\hat{i}(j)} \circ \sigma^i, \hat{i}(j)) \\ &= N^k \circ w \circ \sigma^{\hat{i}(j)} \circ \sigma^i \circ \widehat{\hat{i}(j)} \\ &= N^k \circ w \circ \hat{i} \circ \sigma^j = \psi^k(w, i) \circ \sigma^j \,. \end{aligned}$$

□ *Claim*

This also completes the proof of Proposition 7.2. □

We use Proposition 7.2 to argue that derived semidistributive lattices of the form $\mathbb{C}(L,\gamma)$ are not quotients of L in the most obvious way. It is a standard reasoning to argue that $\delta \in \mathbb{C}(L,\gamma)$ implies $(\delta_0,\delta_1) \in \theta(\gamma_0,\gamma_1)$, where $\theta(\gamma_0,\gamma_1)$ is the congruence generated by the pair (γ_0,γ_1). It is reasonable to ask whether the lattice $\mathbb{C}(L,\gamma)$ is related to the specific quotient lattice $L/\theta(\gamma_0,\gamma_1)$. The following Proposition gives a first answer in the negative, showing that these two lattices are not in general isomorphic.

Proposition 7.5. For $k \in \{\,1,\ldots,n-1\,\}$ the lattice $\mathcal{S}_n/\theta(\perp,\sigma^k)$ is isomorphic to the lattice $\mathcal{S}_k \times \mathcal{S}_{n-k}$.

Proof. We recall that for L a lattice L and θ a congruence of L, each equivalence class $[x]_\theta$ has a least element $\mu_\theta(x)$, computed as follows:

$$\mu_\theta(x) = \bigvee\{\, j \in J(L) \mid j \leq x \text{ and } (j_*,j) \notin \theta \,\}\,.$$

The quotient L/θ is then isomorphic to the poset $\langle\{\,\mu_\theta(x) \mid x \in L\,\},\leq\rangle$. We use this representation to give explicit form to $\mathcal{S}_n/\theta(\perp,\sigma^k)$.

Recall that a permutation is join-irreducible iff it has a unique descent, i.e. a unique index $i \in \{\,1,\ldots,n-1\,\}$ such that $w_i > w_{i+1}$. In [18] we called the pair (w_{i+1},w_i) the principal plan of the join-irreducible w. Let us denote by $\trianglelefteq$ the reflexive transitive closure of the join-dependency relation between join-irreducible permutations. From the characterization given there of the join-dependency relation, we have that $w \trianglelefteq \sigma^k$ iff the $[k,k+1] \subseteq [a,b]$, where a,b is the principal plan of w. Therefore, for w a join-irreducible permutation, we have the following equivalences: $(w_*,w) \in \theta(\perp,\sigma^k)$, iff $w \trianglelefteq \sigma^k$ iff iff $[k,k+1] \subseteq [a,b]$, a,b the principal plan of w. Consequently, a join-irreducible permutation is not congruent to its unique lower cover modulo $\theta(\perp,\sigma^k)$ if and only if its principal plan does not contain the interval $[k,k+1]$.

Remark that, for $a<b<c$, we have $[k,k+1] \subseteq [a,c]$ if and only if $[k,k+1] \subseteq [a,b]$ or $[k,k+1] \subseteq [b,c]$. Using this fact, we see that if $D = D(w)$ is the set of disagreements (or inversions) of some permutation w, then $D' = \{\,(a,b) \in D \mid [k,k+1] \not\subseteq [a,b]\,\}$ is also the set of disagreements of some permutation w'. To this goal, it is enough to verify that D' is closed – i.e. $(a,b),(b,c) \in D'$ implies $(a,c) \in D'$ – and open as well – i.e. $a<b<c$ and $(a,c) \in D'$ implies $(a,b) \in D'$ or $(b,c) \in D'$. Since $(a,b) \in D(w)$ if and only if there exists $j \in J(\mathcal{S}_n)$ such that (a,b) is the principal plan of j, then we deduce that the permutation w' is the least element in the congruence class of w.

Knowing that the order on $\mathcal{S}_n$ is given by inclusion of disagreement sets, the relation

$$D(w') = \{\,(a,b) \in D(w) \mid [a,b] \subseteq [1,k]\,\} \uplus \{\,(a,b) \in D(w) \mid [a,b] \subseteq [k+1,n]\,\}$$

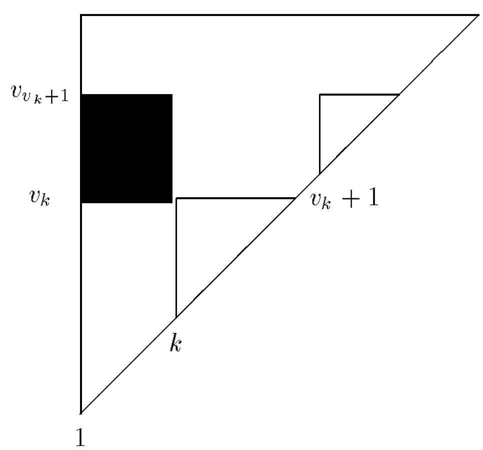
exhibits $\mathcal{S}_n/\theta(\perp,\sigma^k)$ as the order theoretic product of $\mathcal{S}_k$ and $\mathcal{S}_{n-k}$. $\square$

Considering that finite semidistributive lattices and bounded lattices form pseudovarieties [12, 15], we leave it as an open problem whether derived lattices are constructible by means of standard operations such as homomorphic images, subalgebras and products.

We are ready to tackle computation of the lattices derived from Associahedra by atomic covers. The computation we present here is a direct one. Considering however that Associahedra are quotient of Permutohedra, see [17, §9], we expect that the next Proposition may be derived from Proposition 7.2 in a more informative manner.

Proposition 7.6. *Let $\mathcal{T}_n$ be the Associahedron on $n+1$ letters (i.e. the Tamari lattice). If α is an atomic cover of $\mathcal{T}_n$ then $\mathbb{C}(\mathcal{T}_n, \alpha)$ is isomorphic to $\mathcal{T}_{n-1}$.*

To prove Proposition 7.6, we shall review some facts about the explicit representation of the Tamari lattices as lattices of bracketing vectors with the pointwise order, see [11, 2, 5]. A *bracketing vector* is a vector $v \in \{1, \ldots, n\}^n$ such that (i) $i \leq v_i$ and (ii) $i < j \leq v_i$ implies $v_j \leq v_i$. We are going to determine covers of the pointwise order. Let us say that $k \in \{1, \ldots, n-1\}$ is a *split* of a bracketing vector v if $i < k \leq v_i$ implies $v_{v_k+1} \leq v_i$. The next diagram should help understanding the condition: if k is a split, then the black region is a forbidden area meaning that it does not contain points of the form (i, v_i).



Lemma 7.7. *Let k be a split of a bracketing vector v and define the vector v^k by*

$$v_i^k = \begin{cases} v_{v_k+1}, & i = k, \\ v_i, & \textit{otherwise}. \end{cases}$$

Then $v \prec v^k$ and moreover all the covers in $\mathcal{T}_n$ arise in this way.

Proof. We observe first that v^k is again a bracketing vector. Condition (i) is satisfied: if $i \neq k$, then $i \leq v_i = v_i^k$, and otherwise $k \leq v_k < v_k+1 \leq v_{v_k+1} = v_k^k$.

Condition (ii) clearly holds if both i and j are distinct from k. Let us suppose that $j = k$, that is $i < k \leq v_i^k = v_i$. Since k is a split, then $v_k^k = v_{v_k+1} \leq v_i$. Let us suppose that $i = k$, that is, $k < j \leq v_k^k = v_{v_k+1}$. If $j \leq v_k$ then $v_j \leq v_k < v_k^k$. If $v_k < j$, then $v_k + 1 \leq j \leq v_{v_k+1}$ and $v_j \leq v_{v_k+1} = v_k^k$.

Let us suppose that $v < w$ and let k be the least index such that $v_k < w_k$. Observe first that $v_{v_k+1} \leq w_k$: from $v_k < w_k$ we can write $k < v_k + 1 \leq w_k$ and hence $v_{v_k+1} \leq w_{v_k+1} \leq w_k$. Also k is a split of v: if $i < k \leq v_i$ then $i < k \leq w_i$, $w_k \leq w_i$ so that $v_{v_k+1} \leq w_k \leq w_i = v_i$. This shows that $v \prec v^k$ and moreover that any upper cover of v is of the form v^k for some split of v. □

Lemma 7.8. *Let v be a bracketing vector.*

1. *If i, j are two splits of v with $j = v_i + 1$, then we have the following pentagon:*

$$\begin{tikzcd}[column sep=small]
& v_{v_j+1} \dots v_{v_j+1} \dots v_{v_j+1} & \\
v_{v_j+1} \dots v_j \dots v_{v_j+1} \arrow[ur, dash, "j"] & & \\
& & v_i \dots v_{v_j+1} \dots v_{v_j+1} \arrow[uul, dash, "i"'] \\
v_j \dots v_j \dots v_{v_j+1} \arrow[uu, dash, "i"] & & \\
& v_i \dots v_j \dots v_{v_j+1} \arrow[ul, dash, "i"] \arrow[uur, dash, "j"'] &
\end{tikzcd}$$

2. *If i, j are two splits of v and $j \neq v_i + 1$, then we have one of the following diamonds:*

$$\begin{tikzcd}[column sep=small]
& v_{v_i+1} \dots v_{v_i+1} \dots v_{v_j+1} \dots v_{v_j+1} & \\
v_{v_i+1} \dots v_{v_i+1} \dots v_j \dots v_{v_j+1} \arrow[ur, dash, "j"] & & v_i \dots v_{v_i+1} \dots v_{v_j+1} \dots v_{v_j+1} \arrow[ul, dash, "i"'] \\
& v_i \dots v_{v_i+1} \dots v_j \dots v_{v_j+1} \arrow[ul, dash, "i"] \arrow[ur, dash, "j"'] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& v_{v_i+1} \dots v_{v_j+1} \dots v_{v_j+1} \dots v_{v_i+1} & \\
v_{v_i+1} \dots v_j \dots v_{v_j+1} \dots v_{v_i+1} \arrow[ur, dash, "j"] & & v_i \dots v_{v_j+1} \dots v_{v_j+1} \dots v_{v_i+1} \arrow[ul, dash, "i"'] \\
& v_i \dots v_j \dots v_{v_j+1} \dots v_{v_i+1} \arrow[ul, dash, "i"] \arrow[ur, dash, "j"'] &
\end{tikzcd}$$

We refer the reader to [5, Propositions 4 and 5] for a detailed proof of this Lemma.

If v is a bracketing vector and k is a split of v, then we denote the cover $v \prec v^k$ by the pair (v, k). From the previous Lemma it immediately follows:

Corollary 7.9. *If $i \neq j$ and $(v, j) \xrightarrow[v]{} (w, k)$, then $k = j$ and*

$$w = \begin{cases} v^{ii}, & j = v_i + 1, \\ v^i, & \textit{otherwise.} \end{cases}$$

Corollary 7.10. *A cover (v, j) is perspective to the atom $(\perp, k)$ if and only if $j = k$ and $v_k = k$. In this case k is the unique index i such that $v_i = k$.*

Proof. The condition is necessary: the property holds for (v, k) and, by Lemma 7.8, is preserved under the operation of pushing up covers. The condition is also sufficient, for which it is enough to remark that if $v_k = k$, then $\perp \leq v$, $\perp^k = k + 1 \not\leq k = v_k$, $\perp^k = k + 1 \leq v_k^k = v_{v_k+1}$. For the last statement, let us suppose that k is a split of v and that $v_k = k$. If $v_i = k$, then $i \leq k$. However $i < k$ contradicts i being a split. $\square$

We are ready to proof Proposition 7.6.

Proof of Proposition 7.6. From a cover $(v, k) \in \mathbb{C}(\mathcal{T}_n)$, perspective to an atom, define a bracketing vector $\psi(v, k) \in \mathcal{T}_{n-1}$ as the compose

$$\begin{tikzcd} {[n]} \arrow[r, "v"] & {[n]} \arrow[d, "N^k"] \\ {[n-1]} \arrow[u, "\hat{k}"] \arrow[r, "\psi(v,k)"] & {[n-1]} \end{tikzcd} .$$

Before carrying on with the proof, we collect first some remarks. Observe that $N^k(\hat{k}(x)) = x$, while $x \leq \hat{k}(N^k(x))$ and this is an equality if $x \neq k$. Therefore $\hat{k}$ is right adjoint to N^k and moreover N^k is inverse to $\hat{k}$ if restricted to $[n] \setminus \{ k \}$. Also $N^k(x) + 1 = N^k(x + 1)$ if $x \neq k$. If (v, k) is perspective to an atom, so that $v_j = k$ implies $j = k$, an integer of the form $v_{\hat{k}(j)}$ is not equal to k, otherwise $k = \hat{k}(j)$, a contradiction. Consequently we shall use formulas such as $\hat{k}(N^k(v_{\hat{k}(j)})) = v_{\hat{k}(j)}$, and $N^k(v_{\hat{k}(j)}) + 1 = N^k(v_{\hat{k}(j)} + 1)$.

Let us verify that $\psi(v, k)$ is a bracketing vector. The relation $i \leq \psi(v, k)_i = N^k(v_{\hat{k}(i)})$ immediately follows from $\hat{k}(i) \leq v_{\hat{k}(i)}$. If $i < j \leq \psi(v, k)_i = N^k(v_{\hat{k}(i)})$, then $\hat{k}(i) < \hat{k}(j) \leq \hat{k}(N^k(v_{\hat{k}(i)})) = v_{\hat{k}(i)}$ and $v_{\hat{k}(j)} \leq v_{\hat{k}(i)}$ since v is a bracketing vector; the relation $\psi(v, k)_j \leq \psi(v, k)_i$ follows then by applying N^k.

The correspondence ψ is a bijection: given $w \in \mathcal{T}_{n-1}$ the vector $v \in \mathcal{T}_n$, defined by $v_i = k$ if $i = k$ and $v_i = \hat{k}(w_{N^k(i)})$ otherwise, is the unique bracketing vector such that (v, k) is a cover perspective to $(\perp, k)$ and $\psi(v, k) = w$.

We are going to verify that (a) j is a split of $\psi(v, k)$ iff $\hat{k}(j)$ is a split of v, (b) if $(v, k) \xrightarrow[v]{} (w, k)$ then $\psi(w, k) = \psi(v, k)^{N^k(j)}$. From these properties it follows that ψ preserves and reflects the covering relation and therefore it is an order isomorphism.

(a) Let us suppose first that $\hat{k}(j)$ is a split of v and that $l < j \leq \psi(v, i)_l = N^k(v_{\hat{k}(l)})$. It follows that $\hat{k}(l) < \hat{k}(j) \leq \hat{k}(N^k(v_{\hat{k}(l)})) = v_{\hat{k}(l)}$ and therefore

$v_{v_{\hat{k}(j)}+1} \le v_{\hat{k}(l)}$. Hence

$$\begin{aligned}\psi(v,k)_{\psi(v,k)_j+1} &= N^k(v_{\hat{k}(N^k(v_{\hat{k}(j)})+1)}) = N^k(v_{\hat{k}(N^k(v_{\hat{k}(j)}+1))}) \\ &= N^k(v_{v_{\hat{k}(j)}+1}) \le N^k(v_{\hat{k}(l)}) = \psi(v,k)_l \,.\end{aligned}$$

Let us suppose now that j is a split of $\psi(v,k)$ and that $l < \hat{k}(j) \le v_l$. Observe that the relation $l < v_l$ implies that $l \ne k$. Since both l and $\hat{k}(j)$ are distinct from k, the relation $l < \hat{k}(l)$ is strictly preserved by N^k and consequently

$$\begin{aligned}N^k(l) &< N^k(\hat{k}(j)) = j \\ &\le N^k(v_l) = N^k(v_{\hat{k}(N^k(l))}) = \psi(v,k)_{N^k(l)} \,.\end{aligned}$$

We have therefore $\psi(v,k)_{\psi(v,k)_j+1} \le \psi(v,k)_{N^k(l)}$ and

$$\begin{aligned}N^k(v_{v_{\hat{k}(j)}+1}) &= N^k(v_{\hat{k}(N^k(v_{\hat{k}(j)}+1))}) = N^k(v_{\hat{k}(N^k(v_{\hat{k}(j)})+1)}) \\ &= \psi(v,k)_{\psi(v,k)_j+1} \le \psi(v,k)_{N^k(l)} = N^k(v_l) \,.\end{aligned}$$

Transposing this relation and considering that $l \ne k$ we deduce

$$v_{v_{\hat{k}(j)}+1} \le \hat{k}(N^k(v_l)) = v_l \,.$$

(b) Let us suppose that $(v,k) \xrightarrow[v^j]{} (w,k)$, so that $w = v^{jj}$ if $k = v_j + 1$ and $w = v^j$ otherwise. We want to prove that $\psi(w,k) = \psi(v,k)^{N^k(j)}$. Let us begin to show that these two vectors coincide in each component i such that $i \ne N^k(j)$ (or equivalently $j \ne \hat{k}(i)$):

$$\psi(w,k)_i = N^k(w_{\hat{k}(i)}) = N^k(v_{\hat{k}(i)}) = \psi(v,k)_i = \psi(v,k)_i^{N^k(j)} \,.$$

Therefore we are left to compare the values of the two vectors at the coordinate $i = N^k(j)$. On the one hand, we have

$$\begin{aligned}\psi(v,k)^{N^k(j)}_{N^k(j)} &= \psi(v,k)^i_i = \psi(v,k)_{\psi(v,k)_i+1} = N^k(v_{\hat{k}(N^k(v_{\hat{k}(i)})+1)}) \\ &= N^k(v_{\hat{k}(N^k(v_{\hat{k}(i)}+1))}) = N^k(v_{\hat{k}(N^k(v_j+1))}) \\ &= \begin{cases} N^k(v_{k+1}) \,, & k = v_j + 1 \\ N^k(v_{v_j+1}) \,, & \text{otherwise.} \end{cases}\end{aligned}$$

On the other hand, we have

$$\psi(w,k)_{N^k(j)} =$$

$$N^k(w_j) = \begin{cases} \begin{aligned} N^k(v^{jj}_j) &= N^k(v^j_{v^j_j+1}) = N^k(v^j_{v_{v_j+1}+1}) \\ &= N^k(v^j_{v_k+1}) = N^k(v^j_{k+1}) \\ &= N^k(v_{k+1}) \,, \end{aligned} & k = v_j + 1 \,, \\ N^k(v^j_j) = N^k(v_{v_j+1}) \,, & \text{otherwise.} \end{cases}$$

This completes the proof of Proposition 7.6. $\square$

Let us say that a finite semidistributive lattice is *regular* if the lattices $\mathbb{C}(L,\alpha)$, α an atomic cover of L, are all isomorphic. It is not the case that every semidistributive lattice is regular as witnessed for example by the multinomial lattice $\mathcal{L}(2,2,1)$. The reader can find in [2, §6] an introductory discussion of multinomial lattices. Let us recall that, for $v \in \mathbb{N}^k$, the elements of the multinomial lattice $\mathcal{L}(v)$ are words w over an ordered alphabet $\{ a_1, \ldots, a_k \}$ such that, for $i = 1, \ldots, k$, the number $|w|_{a_i}$ of occurrences of the letter a_i equals v_i. The Hasse diagram of $\mathcal{L}(v)$ is obtained by exchanging the position of contiguous letters that appear in the right order. The bottom of the lattice $\mathcal{L}(2,2,1)$ is represented in figure 1. Let $\alpha = aaabbc \prec ababc$ and $\beta = aaabbc \prec aabcb$ be two atoms of this lattice, if we consider the bottoms of $\mathbb{C}(L,\alpha)$ and $\mathbb{C}(L,\beta)$ we observe these two lattices are not isomorphic. We remark that the lattice $\mathcal{L}(2,2,1)$ is not complemented, contrary to the Newman lattices considered in this section. It might be conjectured that complemented semidistributive lattices are regular. More generally it is an open problem to identify sufficient conditions that ensure that a semidistributive lattice is regular.

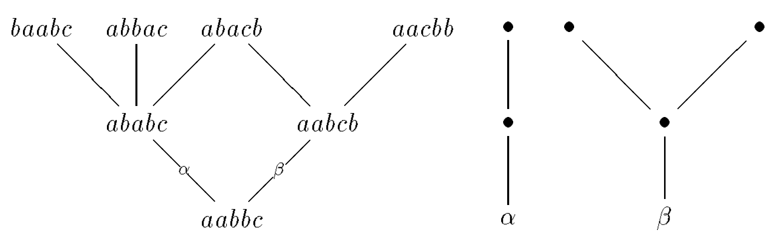


Figure 2: The bottom of the lattices $L = \mathcal{L}(2,2,1)$, $\mathbb{C}(L,\alpha)$ and $\mathbb{C}(L,\beta)$.

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